

Thermoelectric transport in disordered metals without quasiparticles: the SYK models and holography

Richard A. Davison,¹ Wenbo Fu,¹ Antoine Georges,^{2,3,4}

Yingfei Gu,⁵ Kristan Jensen,⁶ and Subir Sachdev^{1,7}

¹*Department of Physics, Harvard University, Cambridge MA 02138, USA*

²*Centre de Physique Théorique, École Polytechnique,
CNRS, 91128 Palaiseau Cedex, France*

³*Collège de France, 11 place Marcelin Berthelot, 75005 Paris, France*

⁴*Department of Quantum Matter Physics, University of Geneva,
24 Quai Ernest-Ansermet, 1211 Geneva 4, Switzerland*

⁵*Department of Physics, Stanford University, Stanford, CA 94305, USA*

⁶*Department of Physics and Astronomy,
San Francisco State University, San Francisco, CA 94132, USA*

⁷*Perimeter Institute for Theoretical Physics,
Waterloo, Ontario, Canada N2L 2Y5*

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Abstract

We compute the thermodynamic properties of the Sachdev-Ye-Kitaev (SYK) models of fermions with a conserved fermion number, Q . We extend a previously proposed Schwarzian effective action to include a phase field, and this describes the low temperature energy and Q fluctuations. We obtain higher-dimensional generalizations of the SYK models which display disordered metallic states without quasiparticle excitations, and we deduce their thermoelectric transport coefficients. We also examine the corresponding properties of Einstein-Maxwell-scalar theories on black brane geometries which interpolate from either AdS_4 or AdS_5 to an $\text{AdS}_2 \times \mathbb{R}^2$ or $\text{AdS}_2 \times \mathbb{R}^3$ near-horizon geometry. These provide holographic descriptions of non-quasiparticle metallic states without momentum conservation. We find a precise match between low temperature transport and thermodynamics of the SYK and holographic models. In both models the Seebeck transport coefficient is exactly equal to the Q -derivative of the entropy. For the SYK models, quantum chaos, as characterized by the butterfly velocity and the Lyapunov rate, universally determines the thermal diffusivity, but not the charge diffusivity.

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I. INTRODUCTION

Strange metal states are ubiquitous in modern quantum materials [1]. Field theories of strange metals [2] have largely focused on disorder-free models of Fermi surfaces coupled to various gapless bosonic excitations which lead to breakdown of the quasiparticle excitations near the Fermi surface, but leave the Fermi surface intact. On the experimental side [1, 3–5], there are numerous indications that disorder effects are important, even though many of the measurements have been performed in nominally clean materials. Theories of strange metals with disorder have only examined the consequences of dilute impurities perturbatively [6–8].

Disordered metallic states have been extensively studied [9] under conditions in which the quasiparticle excitations survive. The quasiparticles are no longer plane wave states as they undergo frequent elastic scattering from impurities, and spatially random and extended quasiparticle states have been shown to be stable under electron-electron interactions [10]. In contrast, the literature on quantum electronic transport is largely silent on the possibility of conducting metallic states without quasiparticle excitations, where the electron-electron scattering length is of order or shorter than the electron-impurity scattering length.

On the other hand, holographic methods do yield many examples of conducting quantum states in the presence of disorder and with no quasiparticle excitations [11–20]. If we assume that the main role of disorder is to dissipate momentum, and we average over disorder to obtain a spatially homogeneous theory, then we may consider homogeneous holographic models which do not conserve momentum. Many such models have been studied [21–31], and their transport properties have been worked out in detail. However, there is not a clear quantum matter interpretation of these disordered, non-quasiparticle, metallic states.

The Sachdev-Ye-Kitaev (SYK) models [32–36] are theories of fermions with a label $i = 1 \dots N$ and random all-to-all interactions. They have many interesting features, including the absence of quasiparticles in a non-trivial, soluble limit in the presence of disorder at low temperature (T). In the limit where N is first taken to infinity and the temperature is subsequently taken to zero, the entropy/ N remains non-zero. Note, however, that such an entropy does not imply an exponentially large ground state degeneracy: it can be achieved by a many-body level spacing

that is of the same order near the ground state as at a typical excited state energy [37]. The SYK models were connected holographically to black holes with AdS_2 horizons, and the $T \rightarrow 0$ limit of the entropy was identified as the Bekenstein-Hawking entropy in Refs. 38–40. Many recent studies have taken a number of perspectives, including the connections to two-dimensional quantum gravity [36, 37, 40–62]. The gravity duals of the SYK models are not in the category of the familiar AdS/CFT correspondence [63], and their low-energy physics is controlled by a symmetry-breaking pattern [47] which also arises in a generic two-derivative theory of dilaton gravity on a nearly AdS_2 spacetime [48–50].

With disordered metallic states in mind, in this paper we will study a class of SYK models [32–35] which have a conserved fermion number¹. The SYK models have recently been extended to lattice models in one or more spatial dimensions [57] (see also [59]), which has opened an exploration into their transport properties. In this work we shall further extend these higher-dimensional models to include a conserved fermion number. This will allow us to describe the thermoelectric response functions of a solvable metallic state without quasiparticles and in the presence of disorder. We will then study thermoelectric transport in holographic theories which have a conserved charge and which break translational symmetry homogeneously via “axion” fields [24–27]. These holographic theories have black brane solutions with $\text{AdS}_2 \times \mathbb{R}^{d-1}$ near-horizon geometries, and we find that their low-temperature thermodynamic and transport properties share precise matches with those of the higher-dimensional SYK models, as we describe below.

The zero-dimensional SYK model of interest to us has canonical complex fermions f_i labeled by $i = 1 \dots N$. We refer to them as the complex SYK models. The Hamiltonian is

$$H_0 = \sum_{\substack{1 \leq i_1 < i_2 \dots i_{q/2} \leq N, \\ 1 \leq i_{q/2+1} < i_{q/2+2} \dots < i_q \leq N}} J_{i_1, i_2 \dots i_q} f_{i_1}^\dagger f_{i_2}^\dagger \dots f_{i_{q/2}}^\dagger f_{i_{q/2+1}} \dots f_{i_{q-1}} f_{i_q}. \quad (1.1)$$

Here q is an even integer, and the couplings $J_{i_1, i_2 \dots i_q}$ are random complex numbers with zero mean obeying

$$\begin{aligned} J_{i_1 i_2 \dots i_{q/2}, i_{q/2+1} \dots i_{q-1} i_q} &= J_{i_{q/2+1} \dots i_{q-1} i_q, i_1 i_2 \dots i_{q/2}}^*, \\ \overline{|J_{i_1, i_2 \dots i_q}|^2} &= \frac{J^2(q/2)!^2}{N^{q-1}}. \end{aligned} \quad (1.2)$$

Note that the case $q = 2$ is special and does have quasiparticles: this describes free fermions and the eigenstates of the random matrix J_{i_1, i_2} obey Wigner-Dyson statistics [64]. Our attention will be focused on $q \geq 4$, and $N \gg q$, when the model flows to a phase without quasiparticles with an

¹ A word about global symmetries is in order. The Majorana SYK model [36] with $2N$ Majorana fermions has a $\text{SO}(2N)$ symmetry only after averaging over disorder. However, this symmetry is not generated by a conserved charge. The model in Eq. (1.1) has a global $\text{U}(1)$ symmetry for each realization of the disorder, and so this symmetry corresponds to a conserved charge. It is this symmetry which is of interest in this work. For completeness, we note that this model acquires an additional $\text{SU}(N)$ symmetry after averaging over disorder.

emergent conformal symmetry at low energies. We define the fermion number $-1/2 < \mathcal{Q} < 1/2$ by

$$\mathcal{Q} = \frac{1}{N} \sum_i \langle f_i^\dagger f_i \rangle - 1/2. \quad (1.3)$$

We also define

$$\Delta = \frac{1}{q}, \quad (1.4)$$

which will be the low-energy scaling dimension of the fermion f .

A. Thermodynamics

We will show in Section II C that the canonical free energy, NF , of the Hamiltonian H_0 in Eq. (1.1) has a low temperature (T) expansion

$$F(\mathcal{Q}, T) = E_0(\mathcal{Q}) - T\mathcal{S}(\mathcal{Q}) + \dots \quad (1.5)$$

where the ground state energy, $E_0(\mathcal{Q})$, is not universal, but the zero-temperature entropy $\mathcal{S}(\mathcal{Q})$ is *universal*, meaning that it depends only on the scaling dimension Δ and is independent of high energy (“UV”) details, such as higher order fermion interactions that could be added to Eq. (1.1). The value of $E_0(\mathcal{Q})$ is not known analytically, but can only be computed numerically or in a large q expansion as we perform in Appendix C. However, remarkably, we can obtain exact results for the universal function $\mathcal{S}(\mathcal{Q})$ for all $0 < \Delta < 1/2$, and for all $-1/2 < \mathcal{Q} < 1/2$ (see Fig. 3). These results agree with those obtained earlier for the special cases $\Delta = 1/4$ and all \mathcal{Q} in Ref. 35, and for $\mathcal{Q} = 0$ and all Δ in Ref. 36. The higher-dimensional complex SYK models also have a free energy of the form Eq. (1.5), where NF is now understood to be the free energy per site of the higher-dimensional lattice.

Because of the non-universality of $E_0(\mathcal{Q})$, the universal properties of the thermodynamics are more subtle in the grand canonical ensemble. The chemical potential, $\mu = (\partial F / \partial \mathcal{Q})_T$, has both universal and non-universal contributions. Consequently it requires a delicate computation to extract the universal portions of the grand potential/ N ,

$$\Omega(\mu, T) = F - \mu \mathcal{Q}. \quad (1.6)$$

It is interesting to note that this universal dependence on \mathcal{Q} , and not μ , is similar to that in the Luttinger theorem for a Fermi liquid: there the Fermi volume is a universal function of \mathcal{Q} , but the connection with μ depends upon many UV details. And indeed, the computation of $\mathcal{S}(\mathcal{Q})$ in Ref. 35 for $\Delta = 1/4$ employs an analysis which parallels that used to prove the Luttinger theorem in Fermi liquids; see also Appendix D.

Section III A will examine in detail the thermodynamics of the simplest holographic axion theory. This theory has a planar black brane solution whose geometry interpolates between AdS_4 near the boundary and $\text{AdS}_2 \times R^2$ near the horizon. The holographic dictionary would suggest that the IR properties of the dual field theory state should be controlled by the near-horizon AdS_2 geometry. We will find that the free energy of the holographic theory also has a low-temperature expansion of the form (1.5), where the universal part of the free energy is determined by the AdS_2 part of the geometry, while the non-universal part depends upon the details of its embedding into the UV AdS_4 geometry. The universal ‘equation of state’, $\mathcal{S}(\mathcal{Q})$ will, however, be different between the SYK and holographic models. The holographic theory we are studying was chosen as it is the simplest theory with momentum dissipation and an AdS_2 horizon – it is not the precise holographic dual of the SYK model.

A quantity that will play a central role in our analyses of the SYK and holographic models is

$$2\pi\mathcal{E} = -\lim_{T \rightarrow 0} \frac{\partial^2 F}{\partial \mathcal{Q} \partial T} = \frac{d\mathcal{S}}{d\mathcal{Q}} = -\lim_{T \rightarrow 0} \left(\frac{\partial \mu}{\partial T} \right)_{\mathcal{Q}}. \quad (1.7)$$

Note that \mathcal{E} is also universal. The factor of 2π has been inserted because then, for theories dual to gravity with an AdS_2 near-horizon geometry, \mathcal{E} is the electric field in the AdS_2 region [40, 65, 66].

B. Effective action

We will also examine aspects of $1/N$ fluctuations about the saddle point which led to the thermodynamic results in Section I A. Here we will follow Ref. 47, who argued that the dominant fluctuations of the Majorana SYK model at low T are controlled by a Schwarzian effective action with $\text{PSL}(2, \mathbb{R})$ time reparameterization symmetry. This effective action can be used to compute energy fluctuations, and hence the specific heat, in the canonical ensemble. In our analysis of the complex SYK model, we find that an additional $\text{U}(1)$ phase field, ϕ , is needed; ϕ is conjugate to \mathcal{Q} fluctuations in the grand canonical ensemble. A similar phase field also appeared in a recent analysis of SYK models with $\mathcal{N} = 2$ supersymmetry [61] with the mean \mathcal{Q} close to zero. We propose in Eq. (2.33) a combined action for energy and \mathcal{Q} fluctuations at a generic mean \mathcal{Q} , with both $\text{PSL}(2, \mathbb{R})$ and $\text{U}(1)$ symmetry. The off-diagonal coupling between energy and \mathcal{Q} fluctuations is controlled by the value of \mathcal{E} . Our effective action will play a central role in the structure of thermoelectric transport, as described in the next subsection.

C. Transport

We will characterize transport by two-point correlators of the conserved number density, which we continue to refer to as \mathcal{Q} , and the conserved energy density $E = H_0/N$. For the zero-dimensional

SYK model in (1.1), both of these quantities are constants of the motion, and so have no interesting dynamics. So we consider here the higher dimensional models, for which \mathcal{Q} and E are defined per site of the higher-dimensional lattice. Then their correlators do have an interesting dependence of wavevector, k , and frequency, ω . We define the dynamic susceptibility matrix, $\chi(k, \omega)$, where

$$\chi(k, \omega) = \begin{pmatrix} \langle \mathcal{Q}; \mathcal{Q} \rangle_{k, \omega} & \langle E - \mu \mathcal{Q}; \mathcal{Q} \rangle_{k, \omega} / T \\ \langle E - \mu \mathcal{Q}; \mathcal{Q} \rangle_{k, \omega} & \langle E - \mu \mathcal{Q}; E - \mu \mathcal{Q} \rangle_{k, \omega} / T \end{pmatrix}, \quad (1.8)$$

and we use the notation

$$\langle A; B \rangle_{k, \omega} \equiv -i \int_0^\infty dt \int d^d x \langle [A(x, t), B(0, 0)] \rangle e^{-ikx + i\omega t}. \quad (1.9)$$

As in the standard analysis of Kadanoff and Martin [67], we expect the low energy and long distance form of these correlators to be fully dictated by the hydrodynamic equations of motion for a diffusive metal [17]. From such an analysis, we obtain, at low frequency and wavenumber,

$$\chi(k, \omega) = [i\omega(-i\omega + Dk^2)^{-1} + 1] \chi_s, \quad (1.10)$$

where D and χ_s are 2×2 matrices. The diffusivities are specified by D , and the static susceptibilities are, as usual, $\chi_s = \lim_{k \rightarrow 0} \lim_{\omega \rightarrow 0} \chi(k, \omega)$. The values of χ_s are related by standard thermodynamic identities to second derivatives of the grand potential Ω , as shown in Eq. (2.41).

One of our main results is that the low T limit of the diffusivity matrix, D , takes a specific form

$$D = \begin{pmatrix} D_1 & 0 \\ 2\pi\mathcal{E}T(D_1 - D_2) & D_2 \end{pmatrix}. \quad (1.11)$$

where D_1 and D_2 are temperature-independent constants. We will find that the result in Eq. (1.11) is obeyed both in the higher-dimensional SYK models, and in the holographic theories. It is a consequence of the interplay between the global $U(1)$ fermion number symmetry and the emergent $\text{PSL}(2, \mathbb{R})$ symmetry of the scaling limit of the SYK model. In holography, $\text{PSL}(2, \mathbb{R})$ is the isometry group of AdS_2 .

We can use the Einstein relation to define a matrix of conductivities

$$\begin{pmatrix} \sigma & \alpha \\ \alpha T & \bar{\kappa} \end{pmatrix} = D\chi_s, \quad (1.12)$$

where σ is the electrical conductivity, α is the thermoelectric conductivity, and $\kappa = \bar{\kappa} - T\alpha^2/\sigma$ is the thermal conductivity. The matrix in Eq. (1.12) is constrained by Onsager reciprocity. From Eqs. (1.11) and (1.12) we find the following result for the low T limit of the thermopower; the Seebeck coefficient S is given in both the SYK and holographic models by

$$\lim_{T \rightarrow 0} S \equiv \frac{\alpha}{\sigma} = \frac{d\mathcal{S}}{d\mathcal{Q}}. \quad (1.13)$$

Since $d\mathcal{S}/d\mathcal{Q} = 2\pi\mathcal{E}$, we see that the Seebeck coefficient is entirely determined by the particle-hole asymmetry of the fermion spectral function.

Eq. (1.13) has been proposed earlier as the ‘Kelvin formula’ by Peterson and Shastry[68] using very different physical arguments. The remarkable aspect of this expression is that it relates a transport quantity to a thermodynamic one, the derivative of the entropy with respect to particle number. Such a relation is in general only approximate, see Ref. 68 and the discussion and applications in Ref 69. Remarkably, this relation holds exactly here: it is an exact consequence of the $\text{PSL}(2, \mathbb{R})$ symmetry of the models under consideration. We note that the form in Eq. (1.11) is implied by Eq. (1.13) and Onsager reciprocity.

We also obtain an interesting result for the Wiedemann-Franz ratio, L , of the SYK model. For the particular higher-dimensional generalization in Eq. (2.45), we find the exact result

$$L \equiv \lim_{T \rightarrow 0} \frac{\kappa}{T\sigma} = \frac{4\pi^2}{3q^2}. \quad (1.14)$$

We comment on aspects of this result:

- For the free fermion case, $q = 2$, this reduces to the universal Fermi liquid Lorenz number $L_0 = \pi^2 k_B^2 / (3e^2)$ (re-inserting fundamental constants). Although expected, this agreement with L_0 at $q = 2$ is non-trivial and remarkable: rather than the usual arguments based upon integrals over the Fermi function, Eq. (1.14) arises from the structure of *bosonic* normal modes of the $1/N$ fluctuations, as discussed in Appendix F.
- The decrease of L for large q can be understood as follows. As we will see in the large q solution in Appendix C, the energy bandwidth for fermion states vanishes as $q \rightarrow \infty$. Consequently, fermion hopping transfers little energy, and the thermal conductivity κ is suppressed. In contrast, fermion hopping continues to transfer unit charge, and hence the conductivity does not have a corresponding suppression.
- Although the result in Eq. (1.14) appears universal, it is not so: there are other higher-dimensional generalizations of the inter-site coupling term in Eq. (2.45) which will lead to corrections to the value of L ; see Section II E. Needless to say, these corrections will not modify the universal value L_0 at $q = 2$. The non-universality of L for higher q is connected to the non-renormalization of inter-site disorder in the present large N limit.
- Results for the values of L for the holographic models appear in the body of the paper.

Note: While we were completing this work we were made aware of Ref. [70], which has some overlap with our holographic analysis.

II. COMPLEX SYK MODEL

A. Large N saddle point

In this section, we employ Green's functions in the grand canonical ensemble at a chemical potential μ . Starting from a perturbative expansion of H_0 in Eq. (1.1), and averaging term-by-term, we obtain the following equations for the Green's function and self energy in the large N limit:

$$\Sigma(\tau) = -(-1)^{q/2} q J^2 [G(\tau)]^{q/2} [G(-\tau)]^{q/2-1} \quad (2.1)$$

$$G(i\omega_n) = \frac{1}{i\omega_n + \mu - \Sigma(i\omega_n)}; \quad (2.2)$$

where ω_n is a Matsubara frequency $\omega_n = 2\pi(n + \frac{1}{2})$ and τ is imaginary time. As in Refs. 32 and 35, we make the following IR ansatz at a complex frequency z

$$G(z) = C \frac{e^{-i(\pi\Delta+\theta)}}{z^{1-2\Delta}} \quad , \quad \text{Im}(z) > 0, \quad (2.3)$$

which is expressed in terms of three real parameters, C , Δ and θ . Here the complex frequency is small compared to the disorder, $|z| \ll J$. As we describe in Appendix B, this is the appropriate form for the two-point function of a charged operator at nonzero chemical potential in a limit where there is an approximate conformal invariance. Unitarity implies that the spectral weight is positive, which in turn implies that

$$-\pi\Delta < \theta < \pi\Delta. \quad (2.4)$$

The particle-hole symmetric value is $\theta = 0$. Inserting Eq. (2.3) into (2.2), a straightforward analysis described in Appendix A shows that Δ is given by (1.4), while

$$C = \left[\frac{\Gamma(2(1-1/q))}{\pi q J^2} \right]^{1/q} \left[\frac{\pi}{\Gamma(2/q)} \right]^{1-1/q} [\sin(\pi/q + \theta) \sin(\pi/q - \theta)]^{1/q-1/2}. \quad (2.5)$$

The value of θ remains undetermined in this IR analysis. Below, in Eq. (2.24), we find an exact relationship between θ and the density \mathcal{Q} , as was first found in Ref. 35 for the $\Delta = 1/4$ theory.

B. Non-zero temperature

Fourier transforming the fermion Green's function Eq. (2.3) gives

$$G(\tau) \sim \begin{cases} -|\tau|^{-2\Delta}, & \tau > 0, \\ e^{-2\pi\mathcal{E}} |\tau|^{-2\Delta}, & \tau < 0, \end{cases} \quad (2.6)$$

where the “spectral asymmetry” \mathcal{E} is related to θ as

$$e^{2\pi\mathcal{E}} = \frac{\sin(\pi\Delta + \theta)}{\sin(\pi\Delta - \theta)}. \quad (2.7)$$

The asymmetry in (2.6) was argued in [71] (and reviewed in [40]) to fix the T derivative of μ

$$\lim_{T \rightarrow 0} \left(\frac{\partial \mu}{\partial T} \right)_{\mathcal{Q}} = -2\pi\mathcal{E}. \quad (2.8)$$

So this \mathcal{E} is the same as that introduced in Eq. (1.7). See Appendix B for an independent argument for this relation from low-energy conformal symmetry.

The $T > 0$ generalization of Eq. (2.6) is a saddle point of the action in Eq. (C14)

$$G_s(\tau) = -C \left(\frac{\Gamma(2\Delta) \sin(\pi\Delta + \theta)}{\pi} \right) e^{-2\pi\mathcal{E}T\tau} \left(\frac{\pi T}{\sin(\pi T\tau)} \right)^{2\Delta}, \quad 0 < \tau < \frac{1}{T}, \quad (2.9)$$

with \mathcal{E} remaining independent of T provided \mathcal{Q} is held fixed. This result was found both in [40, 71], and in the AdS_2 computation in [72]. After using the KMS condition and (2.7), the $T \rightarrow 0$ limit of (2.9) agrees with (A7).

C. Thermodynamics

Ref. 47 has given an expression for the free energy of the Majorana SYK models as a functional of the Green’s function and the self energy; related expressions were given earlier for the complex SYK models [35, 40]. It is straightforward to obtain a similar result for the grand potential of the complex SYK models, which we give in Eq. (C14). Here, we will only compute the Δ derivative of the grand potential in the $T \rightarrow 0$ limit, and then integrate with respect to Δ to obtain the low-temperature grand potential.

The only term in the grand potential which explicitly depends on q is (see Eq. (C14))

$$\Omega = \dots - J^2 \int_0^{1/T} d\tau [G(\tau)]^{q/2} [G(1/T - \tau)]^{q/2}. \quad (2.10)$$

Substituting $\Delta = 1/q$ and using (2.5), (2.7) and (2.9) the leading low-temperature derivative of Ω with respect to Δ comes from this term,

$$\begin{aligned} \frac{d\Omega}{d\Delta} &= -\frac{2(2\Delta - 1)}{\pi} \frac{\sin(\pi\Delta + \theta) \sin(\pi\Delta - \theta)}{\sin(2\pi\Delta)} \int_0^{1/T} d\tau \left(\frac{\pi T}{\sin(\pi T\tau)} \right)^2 \ln \left(\frac{\Lambda}{\sin(\pi T\tau)} \right) \\ &= -2\pi T(2\Delta - 1) \frac{\sin(\pi\Delta + \theta) \sin(\pi\Delta - \theta)}{\sin(2\pi\Delta)} + \text{a term of order } J, \end{aligned} \quad (2.11)$$

where Λ is some τ -independent constant. We subtract the term in Ω of order J , which we call Ω_0 . Using Eq. (2.7) to express the result in terms of \mathcal{E} we obtain

$$\begin{aligned}\frac{d(\Omega - \Omega_0)}{d\Delta} &= -\pi T(2\Delta - 1) \frac{\sin(2\pi\Delta)}{\cos(2\pi\Delta) + \cosh(2\mathcal{E}\pi)} \\ &= -\frac{\pi T(2\Delta - 1)}{2} [\tan(\pi(\Delta - i\mathcal{E})) + \tan(\pi(\Delta + i\mathcal{E}))].\end{aligned}\tag{2.12}$$

The relation (1.7) between \mathcal{E} , μ , and T , implies that \mathcal{E} is fixed when μ and T are also fixed. Thus in writing $d\Omega/d\Delta$ we treat \mathcal{E} and Δ as independent variables, in particular, we keep \mathcal{E} fixed when integrating over Δ . Note that the relationship between \mathcal{E} and \mathcal{Q} in (2.23) depends upon q , and so varying q at fixed \mathcal{E} implies that \mathcal{Q} will vary. We can now integrate (2.12) to obtain

$$\begin{aligned}\Omega - \Omega_0 &= \frac{(2\Delta - 1)T}{2} \ln [2(\cos(2\pi\Delta) + \cosh(2\pi\mathcal{E}))] \\ &\quad + \frac{iT}{4\pi} [\text{Li}_2(-e^{2\pi(\mathcal{E}-i\Delta)}) + \text{Li}_2(-e^{2\pi(-\mathcal{E}-i\Delta)}) - \text{Li}_2(-e^{2\pi(\mathcal{E}+i\Delta)}) - \text{Li}_2(-e^{2\pi(-\mathcal{E}+i\Delta)})] \\ &\equiv -T\mathcal{G}(\mathcal{E}).\end{aligned}\tag{2.13}$$

The integration constant is fixed by the boundary conditions that the singular part of the grand potential, $\Omega - \Omega_0$ vanishes at the free fermion point $\Delta = 1/2$. The last line in Eq. (2.13) defines the function $\mathcal{G}(\mathcal{E})$.

1. Separating the universal and non-universal parts

The grand potential Ω computed in (2.13) depends upon \mathcal{E} and T . But, in the grand canonical ensemble at fixed μ and T , \mathcal{E} has an unknown dependence upon μ and T . It is therefore better to convert to the canonical ensemble at fixed \mathcal{Q} and T , where we know the \mathcal{Q} and T dependence of μ from (2.8),

$$\mu(\mathcal{Q}, T) = \mu_0 - 2\pi\mathcal{E}(\mathcal{Q})T + \dots\tag{2.14}$$

as $T \rightarrow 0$. Here \mathcal{E} depends only on \mathcal{Q} , and μ_0 is the contribution from the ground state energy, E_0 , with

$$\mu_0 = \frac{dE_0}{d\mathcal{Q}}.\tag{2.15}$$

The complete grand potential, including the contribution of the ground state energy, is

$$\Omega = E_0 - \mu_0\mathcal{Q} - T\mathcal{G}(\mathcal{E}) + \dots,\tag{2.16}$$

where the functional form of the singular term $\mathcal{G}(\mathcal{E})$ was given in (2.13). The free energy in the canonical ensemble, F , is

$$F(\mathcal{Q}, T) = \Omega + \mu\mathcal{Q}.\tag{2.17}$$

Now we use the thermodynamic identity

$$\mu = \left(\frac{\partial F}{\partial Q} \right)_T, \quad (2.18)$$

to obtain an expression for the density, \mathcal{Q} . Using (2.14-2.17), (2.18) becomes

$$\mu_0 - 2\pi\mathcal{E}T = \frac{dE_0}{dQ} - T \frac{d\mathcal{G}}{d\mathcal{E}} \frac{d\mathcal{E}}{dQ} - 2\pi\mathcal{E}T - 2\pi T \frac{d\mathcal{E}}{dQ} \mathcal{Q}, \quad (2.19)$$

which gives us

$$\mathcal{Q} = -\frac{1}{2\pi} \frac{d\mathcal{G}}{d\mathcal{E}}. \quad (2.20)$$

Similarly, the entropy is

$$\begin{aligned} \mathcal{S} &= - \left(\frac{\partial F}{\partial T} \right)_{\mathcal{Q}} \\ &= \mathcal{G} + 2\pi\mathcal{E}\mathcal{Q} \end{aligned} \quad (2.21)$$

Eqs. (2.20) and (2.21) show that $\mathcal{S}(\mathcal{Q})$ and $\mathcal{G}(\mathcal{E})$ are a Legendre pair, and so

$$\frac{d\mathcal{S}}{d\mathcal{Q}} = 2\pi\mathcal{E}. \quad (2.22)$$

This equality is equivalent to Eq. (2.8) by the Maxwell relation in Eq. (1.7), and this supports the validity of our analysis.

Appendix C presents a computation of the thermodynamics at large q . In this limit, we explicitly verify the above decompositions into universal and non-universal components.

2. Charge

We compute the density from Eqs. (2.13) and (2.20) to obtain

$$\mathcal{Q} = \frac{(2\Delta - 1) \sinh(2\pi\mathcal{E})}{2(\cos(2\pi\Delta) + \cosh(2\pi\mathcal{E}))} - \frac{i}{4\pi} \ln \left[\frac{(1 + e^{2\pi(\mathcal{E}-i\Delta)})(1 + e^{2\pi(-\mathcal{E}+i\Delta)})}{(1 + e^{2\pi(\mathcal{E}+i\Delta)})(1 + e^{2\pi(-\mathcal{E}-i\Delta)})} \right]. \quad (2.23)$$

This simplifies considerably when expressed in terms of θ via (2.7)

$$\mathcal{Q} = -\frac{\theta}{\pi} + \left(\Delta - \frac{1}{2} \right) \frac{\sin(2\theta)}{\sin(2\pi\Delta)}. \quad (2.24)$$

This agrees with Appendix A of Ref. 35 at $q = 4$. In Appendix D, we generalize the Luttinger-Ward argument of Ref. 35 to arbitrary q , and provide further evidence for Eq. (2.24). Note that $\mathcal{Q} = \pm 1/2$ at the limiting values $\theta = \mp \pi\Delta$. A plot of the density appears in Fig. 1.

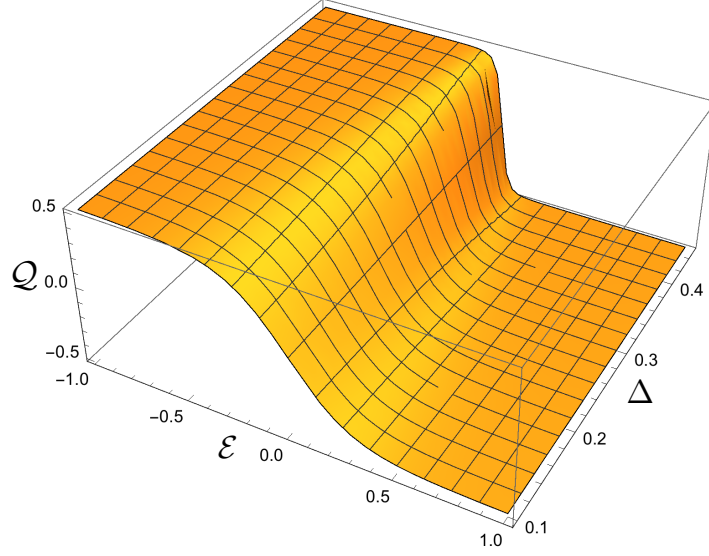


FIG. 1. The density \mathcal{Q} as a function of \mathcal{E} and Δ

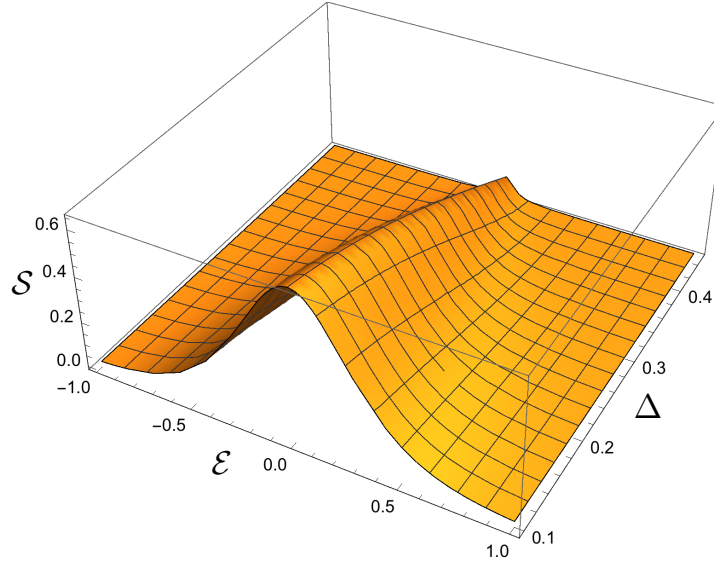


FIG. 2. The entropy \mathcal{S} as a function of \mathcal{E} and Δ

3. Entropy

We can compute the entropy \mathcal{S} from (2.13), (2.21) and (2.23). It can be verified that $\mathcal{S} \rightarrow 0$ as $\mathcal{E} \rightarrow \pm\infty$. A plot of the entropy as a function of \mathcal{E} appears in Fig. 2. We can combine Figs. 1 and 2 to obtain the entropy as a function of density, and this is shown in Fig. 3. In Appendix E, we present the results of the numerical solution of the saddle point equations in Eqs. (2.1) and (2.2)

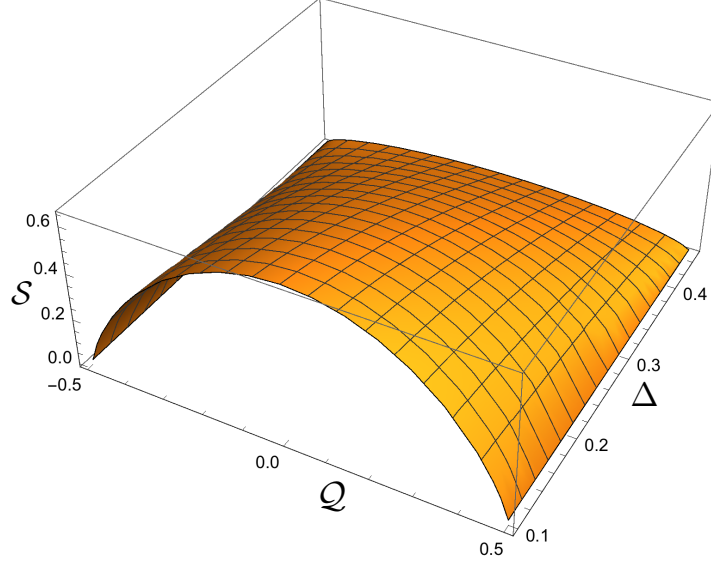


FIG. 3. The entropy \mathcal{S} as a function of \mathcal{Q} and Δ

for $q = 4$, and find good agreement with the analytic results above.

At the particle-hole symmetric point, $\mathcal{E} = \mathcal{Q} = 0$, this yields from (2.13)

$$\mathcal{S}(0) = \frac{(1 - 2\Delta)}{2} \ln [4 \cos^2(\pi\Delta)] - \frac{i}{2\pi} [\text{Li}_2(-e^{-2\pi i\Delta}) - \text{Li}_2(-e^{2\pi i\Delta})], \quad (2.25)$$

which agrees with Kitaev's result [36].

D. Fluctuations

This subsection presents an analysis of the “zero mode” fluctuations about the large N saddle point found above. We will generalize the Schwarzian effective action, proposed in Ref. 47, to non-zero μ , and relate its coupling constants to thermodynamic derivatives.

While solving the equations for the Green's function and the self energy, Eqs. (2.1) and (2.2), we found that, at $\omega, T \ll J$, the $i\omega + \mu$ term in the inverse Green's function could be ignored in determining the IR solution Eq. (2.3). After dropping the $i\omega + \mu$ term, it is not difficult to show that Eqs. (2.1) and (2.2) have remarkable, emergent, time reparameterization and U(1) invariances. This is clearest if we write the Green's function in a two-time notation, i.e. $G(\tau_1, \tau_2)$; then Eqs. (2.1) and (2.2) are invariant under [36, 40]

$$\begin{aligned} G(\tau_1, \tau_2) &= [f'(\tau_1)f'(\tau_2)]^\Delta \frac{g(\tau_2)}{g(\tau_1)} G(f(\tau_1), f(\tau_2)) \\ \Sigma(\tau_1, \tau_2) &= [f'(\tau_1)f'(\tau_2)]^{1-\Delta} \frac{g(\tau_1)}{g(\tau_2)} \Sigma(f(\tau_1), f(\tau_2)) \end{aligned} \quad (2.26)$$

where $f(\tau)$ and $g(\tau)$ are arbitrary functions representing the reparameterizations of time and $U(1)$ transformations respectively.

Next, we observe that these approximate symmetries are broken by the saddle-point solution, G_s in Eq. (2.9). So, following Ref. 47, we deduce an effective action for the associated Nambu-Goldstone modes by examining the action of the symmetries on the saddle-point solution,

$$G(\tau_1, \tau_2) = [f'(\tau_1)f'(\tau_2)]^\Delta G_s(f(\tau_1), f(\tau_2))e^{i\phi(\tau_1)-i\phi(\tau_2)}, \quad (2.27)$$

Here, we find it convenient to parameterize $g(\tau) = e^{-i\phi(\tau)}$ in terms of a phase field ϕ . We will soon see that its derivative is conjugate to density fluctuations. Our remaining task [47] is to (i) find the set of $f(\tau)$ and $\phi(\tau)$ which leave Eq. (2.27) invariant *i.e.* Eq. (2.27) holds after we replace the l.h.s. by $G_s(\tau_1, \tau_2)$; and (ii) propose an effective action which has the property of remaining invariant under the set of $f(\tau)$ and $\phi(\tau)$ which leave Eq. (2.27) invariant.

For the first task, we find [36, 47] that only reparameterizations, $f(\tau)$ belonging to $\text{PSL}(2, \mathbb{R})$ leave Eq. (2.27) invariant. At $T > 0$, we need $\text{PSL}(2, \mathbb{R})$ transformations which map the thermal circle $0 < \tau < 1/T$ to itself. These are given by

$$\frac{1}{\pi T} \tan(\pi T f(\tau)) = \frac{a \tan(\pi T \tau) + b\pi T}{c \tan(\pi T \tau) + d\pi T} \quad , \quad ad - bc = 1, \quad (2.28)$$

where a, b, c, d are real numbers. This transformation is more conveniently written in terms of unimodular complex numbers

$$z = e^{2\pi i T \tau} \quad , \quad z_f = e^{2\pi i T f(\tau)} \quad (2.29)$$

as

$$z_f = \frac{w_1 z + w_2}{w_2^* z + w_1^*} \quad , \quad |w_1|^2 - |w_2|^2 = 1, \quad (2.30)$$

where $w_{1,2}$ are complex numbers. Applying Eq. (2.30) to Eqs. (2.9) and (2.27), we find that Eq. (2.27) remains invariant only for the particle-hole symmetric case $\mathcal{E} = 0$, which was considered previously [47]. However, Eq. (2.27) is invariant under $\text{PSL}(2, \mathbb{R})$ transformations when the phase field $\phi(\tau)$ is related to the $\text{PSL}(2, \mathbb{R})$ transformation $f(\tau)$ as

$$-i\phi(\tau) = 2\pi \mathcal{E} T (\tau - f(\tau)) \quad (2.31)$$

The effective action is required to vanish when $\phi(\tau)$ satisfies Eq. (2.31): this is the key result of this subsection, and is the origin of the constraints on thermoelectric properties described in this paper.

Now we can turn to the second task of obtaining an effective for $f(\tau)$ and $\phi(\tau)$ which is invariant Eqs. (2.30) and (2.31). It is more convenient to use the parameterization

$$f(\tau) \equiv \tau + \epsilon(\tau), \quad (2.32)$$

and express the action in terms $\phi(\tau)$ and $\epsilon(\tau)$. Generalizing the reasoning in Ref. 47, we propose the action

$$\frac{S_{\phi,\epsilon}}{N} = \frac{K}{2} \int_0^{1/T} d\tau [\partial_\tau \phi + i(2\pi\mathcal{E}T)\partial_\tau \epsilon]^2 - \frac{\gamma}{4\pi^2} \int_0^{1/T} d\tau \{\tan(\pi T(\tau + \epsilon(\tau))), \tau\}. \quad (2.33)$$

Higher powers of the first term in square brackets can also be present, but we do not consider them here. The curly brackets in the second term represent a Schwarzian derivative

$$\{f, \tau\} \equiv \frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'} \right)^2, \quad (2.34)$$

which has the important property of vanishing under $\text{PSL}(2, \mathbb{R})$ transformations.

Our reasoning above falls short of a complete derivation of the structure of the effective action in Eq. (2.33). The missing ingredient is our assumption that it is permissible to expand the action in gradients of ϕ and ϵ , when the saddle-point action contains long-range power-law interactions in time. If this assumption was not valid, then the phenomenological couplings K and γ would diverge in the $T \rightarrow 0$ limit. We compute the values of K and γ in Appendix C using a large q expansion and find that they are finite as $T \rightarrow 0$. This a posteriori justifies our gradient expansion. We also present a normal-mode analysis of fluctuations of the underlying path integral for the SYK model in Appendix F; this follows the analysis of Ref. 47, and uses their reasoning to provide an alternative motivation of Eq. (2.33).

We now relate the phenomenological couplings, γ and K , to thermodynamic quantities by computing the fluctuations of energy and number density implied by $S_{\phi,\epsilon}$ in the large N limit. The energy and density operators are defined by

$$\delta E(\tau) - \mu \delta \mathcal{Q}(\tau) = \frac{1}{N} \frac{\delta S_{\phi,\epsilon}}{\delta \epsilon'(\tau)} \quad , \quad \delta \mathcal{Q}(\tau) = \frac{i}{N} \frac{\delta S_{\phi,\epsilon}}{\delta \phi'(\tau)}. \quad (2.35)$$

Introducing,

$$\tilde{\phi}(\tau) = \phi(\tau) + i2\pi\mathcal{E}T\epsilon(\tau) \quad (2.36)$$

and expanding (2.33) to quadratic order in ϕ and ϵ , we obtain the Gaussian action

$$\frac{S_{\phi,\epsilon}}{N} = \frac{KT}{2} \sum_{\omega_n \neq 0} \omega_n^2 \left| \tilde{\phi}(\omega_n) \right|^2 + \frac{T\gamma}{8\pi^2} \sum_{|\omega_n| \neq 0, 2\pi T} \omega_n^2 (\omega_n^2 - 4\pi^2 T^2) |\epsilon(\omega_n)|^2 + \dots \quad (2.37)$$

where ω_n is a Matsubara frequency. Note the restrictions on $n = 0, \pm 1$ frequencies in (2.37), which are needed to eliminate the zero modes associated with $\text{PSL}(2, \mathbb{R})$ and $\text{U}(1)$ invariances. In terms of $\tilde{\phi}(\tau)$ and $\epsilon(\tau)$, Eq. (2.35) is

$$\begin{aligned} \delta \mathcal{Q}(\tau) &= iK \tilde{\phi}'(\tau) \\ \delta E(\tau) - \mu_0 \delta \mathcal{Q}(\tau) &= -\frac{\gamma}{4\pi^2} [\epsilon'''(\tau) + 4\pi^2 T^2 \epsilon'(\tau)] + i2\pi K \mathcal{E} \tilde{\phi}'(\tau). \end{aligned} \quad (2.38)$$

Now we compute the correlators of these observables in the Gaussian action in Eq. (2.37), following the methods of Ref. 47. We have for the two-point correlator of $\tilde{\phi}(\tau)$

$$\begin{aligned}\langle \tilde{\phi}(\tau)\tilde{\phi}(0) \rangle &= \frac{T}{NK} \sum_{\omega_n \neq 0} \frac{e^{i\omega_n \tau}}{\omega_n^2} \\ &= \frac{1}{NKT} \left[\frac{1}{2} \left(T\tau - \frac{1}{2} \right)^2 - \frac{1}{24} \right] \quad \text{for } 0 < T\tau < 1,\end{aligned}\tag{2.39}$$

and extended periodically for all τ with period $1/T$. Similar for $\epsilon(\tau)$

$$\begin{aligned}\langle \epsilon(\tau)\epsilon(0) \rangle &= \frac{4\pi^2 T}{N\gamma} \sum_{|\omega_n| \neq 0, 2\pi T} \frac{e^{i\omega_n \tau}}{\omega_n^2(\omega_n^2 - 4\pi^2 T^2)} \\ &= \frac{1}{N\gamma T^3} \left[\frac{1}{24} + \frac{1}{4\pi^2} - \frac{1}{2} \left(T\tau - \frac{1}{2} \right)^2 + \frac{5}{8\pi^2} \cos(2\pi T\tau) + \frac{1}{2\pi} \left(T\tau - \frac{1}{2} \right) \sin(2\pi T\tau) \right] \\ &\quad \text{for } 0 < T\tau < 1.\end{aligned}\tag{2.40}$$

Inserting Eqs. (2.39) and (2.40) into Eq. (2.38), we confirm that the correlators of the conserved densities are τ -independent; their second moment correlators, which define the matrix of static susceptibility correlators by (1.8), are given by

$$\begin{aligned}\chi_s &= \frac{1}{N} \begin{pmatrix} -(\partial^2 \Omega / \partial \mu^2)_T & -(\partial^2 \Omega / \partial \mu \partial T)_\mu \\ -T(\partial^2 \Omega / \partial \mu \partial T)_\mu & -T(\partial^2 \Omega / \partial T^2)_\mu \end{pmatrix} \\ &= \frac{1}{T} \begin{pmatrix} \langle (\delta \mathcal{Q})^2 \rangle & \langle (\delta E - \mu \delta \mathcal{Q}) \delta \mathcal{Q} \rangle / T \\ \langle (\delta E - \mu \delta \mathcal{Q}) \delta \mathcal{Q} \rangle & \langle (\delta E - \mu \delta \mathcal{Q})^2 \rangle / T \end{pmatrix} \\ &= \frac{1}{N} \begin{pmatrix} K & 2\pi K \mathcal{E} \\ 2\pi K \mathcal{E} T & (\gamma + 4\pi^2 \mathcal{E}^2 K) T \end{pmatrix}\end{aligned}\tag{2.41}$$

From Eq. (2.41) we obtain the relationship between the couplings K and γ in the effective action in Eq. (2.33). After application of some thermodynamic identities, we can write these as

$$K = \left(\frac{\partial \mathcal{Q}}{\partial \mu} \right)_T, \quad \gamma = - \left(\frac{\partial^2 F}{\partial T^2} \right)_\mathcal{Q},\tag{2.42}$$

and also confirm the thermodynamic definitions of \mathcal{E} in Eqs. (1.7), (2.8), and (2.22).

Appendix G presents another argument for the results in Eq. (2.42) without computation of fluctuations of the effective action.

E. Higher-dimensional SYK theory

Gu *et al.* have defined a set of higher-dimensional SYK models [57] for Majorana fermions and computed their energy transport properties. Here we extend their results to the case of complex fermions at a general μ , and discuss their thermoelectric transport.

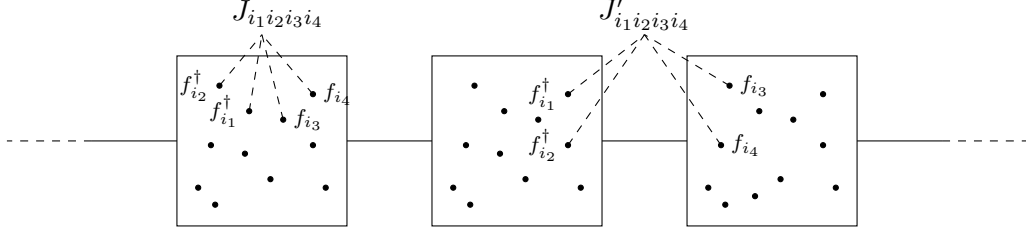


FIG. 4. A chain of coupled SYK sites with complex fermions (in this figure we draw $q = 4$ case): each site contains $N \gg 1$ fermions with on-site interactions as in (1.1). The coupling between nearest neighbor sites are four fermion interaction with two from each site. In general, one can consider other types of q -body interactions ($q = 4$ in this caption), e.g. $f_{x,i_1}^\dagger f_{x,i_2} f_{x+1,i_3}^\dagger f_{x+1,i_4}$. Such terms will only change the ratio between D_1 and D_2 , i.e. Eq. (2.48) by a non-universal coefficient depends on the details of the model. In particular, if we only have $f_{x,i_1}^\dagger f_{x,i_2} f_{x+1,i_3}^\dagger f_{x+1,i_4}$ -type terms to couple the nearest neighbour sites, charge diffusion D_1 will vanish due to the local charge conservation.

We will limit our presentation to one spatial dimension (although the results easily generalize to all spatial dimensions). We consider the model

$$H = \sum_x (H_x + \delta H_x) \quad (2.43)$$

The on-site term H_x is equivalent to a copy of Eq. (1.1) on each site x

$$H_x = \sum_{\substack{1 \leq i_1 < i_2 < \dots < i_{q/2} \leq N, \\ 1 \leq i_{q/2+1} < i_{q/2+2} < \dots < i_q \leq N}} J_{x,i_1,i_2,\dots,i_q} f_{x,i_1}^\dagger \dots f_{x,i_{q/2}}^\dagger f_{x,i_{q/2+1}} \dots f_{x,i_q} \quad (2.44)$$

The nearest neighbor coupling term δH_x denotes nearest-neighbor interactions as shown in Fig 4,

$$\delta H_x = \sum_{\substack{1 \leq i_1 < i_2 < \dots < i_{q/2} \leq N, \\ 1 \leq i_{q/2+1} < i_{q/2+2} < \dots < i_q \leq N}} J'_{x,i_1,i_2,\dots,i_q} f_{x,i_1}^\dagger \dots f_{x,i_{q/2}}^\dagger f_{x+1,i_{q/2+1}} \dots f_{x+1,i_q} + \text{H.c.} \quad (2.45)$$

The couplings $\{J_{x,i_1,i_2,\dots,i_q}\}$ and $\{J'_{x,i_1,i_2,\dots,i_q}\}$ are all independent random variables² with zero mean, and variances given by

$$\overline{|J_{x,i_1,i_2,\dots,i_q}|^2} = \frac{J_0^2 (q/2)!^2}{N^{q-1}}, \quad \overline{|J'_{x,i_1,i_2,\dots,i_q}|^2} = \frac{J_1^2 (q/2)!^2}{N^{q-1}}. \quad (2.46)$$

We remark that the particular interaction we choose here is just one possible example, and this particular choice produces the Wiedemann-Franz ratio discussed in Eq. (1.14). In general, we

² except that J_{x,i_1,i_2,\dots,i_q} need to satisfy the hermitian condition as shown in Eq. (1.2).

can choose p fermions from one site (with p_1 many f^\dagger and p_2 many f , p_1 and p_2 can be chosen arbitrarily) to couple $(q - p)$ fermions in the nearest neighbor site (with $(q/2 - p_1)$ many f^\dagger and $(q/2 - p_2)$ many f). For example, for $q = 4$ case, we are allowed to have following couplings between x and $x + 1$:

$$\begin{aligned} p = 1 : & \quad f_{x,i_1}^\dagger f_{x+1,i_2}^\dagger f_{x+1,i_3} f_{x+1,i_4}, \quad f_{x,i_1} f_{x+1,i_2}^\dagger f_{x+1,i_3}^\dagger f_{x+1,i_4}; \\ p = 2 : & \quad f_{x,i_1}^\dagger f_{x,i_2}^\dagger f_{x+1,i_3} f_{x+1,i_4}, \quad f_{x,i_1}^\dagger f_{x,i_2} f_{x+1,i_3}^\dagger f_{x+1,i_4}, \quad f_{x,i_1} f_{x,i_2} f_{x+1,i_3}^\dagger f_{x+1,i_4}; \\ p = 3 : & \quad f_{x,i_1}^\dagger f_{x,i_2}^\dagger f_{x,i_3} f_{x+1,i_4}, \quad f_{x,i_1}^\dagger f_{x,i_2} f_{x,i_3} f_{x+1,i_4}; \end{aligned}$$

If we include these couplings with different coefficient, the Wiedemann-Franz ratio will be non-universal. However, for $q = 2$ model, the only term we can add is $f_{i_1,x}^\dagger f_{i_2,x+1}$ and therefore the Wiedemann-Franz ratio goes back to $\pi^2/3$ as discussed previously.

Following the analysis in Ref. 57, the effective action for the higher dimensional model can be deduced from that of the zero dimensional model. Using the results in Appendices F and H, we find that the Gaussian action for energy and density fluctuations in higher dimensions generalizes from Eq. (2.37) to

$$\begin{aligned} \frac{S_{\phi,\epsilon}}{N} = & \frac{KT}{2} \sum_{k,\omega_n \neq 0} |\omega_n| (D_1 k^2 + |\omega_n|) \left| \tilde{\phi}(k, \omega_n) \right|^2, \\ & + \frac{T\gamma}{8\pi^2} \sum_{k, |\omega_n| \neq 0, 2\pi T} |\omega_n| (D_2 k^2 + |\omega_n|) (\omega_n^2 - 4\pi^2 T^2) |\epsilon(k, \omega_n)|^2, \end{aligned} \quad (2.47)$$

where D_1 and D_2 are the diffusion constants of the conserved charges. In Appendix H we find that their ratio obeys

$$\frac{D_2}{D_1} = \frac{4\pi^2 \Delta^2 K}{3 \gamma}. \quad (2.48)$$

Following Ref. 57, from Eq. (2.47), and including a contact term as described in Ref. 73, we can obtain the long-wavelength and low frequency dynamic susceptibilities

$$\begin{aligned} \langle \mathcal{Q}; \mathcal{Q} \rangle_{k,\omega} &= K \frac{D_1 k^2}{-i\omega + D_1 k^2}, \\ \langle E - \mu \mathcal{Q}; \mathcal{Q} \rangle_{k,\omega} &= 2\pi K \mathcal{E} T \frac{D_1 k^2}{-i\omega + D_1 k^2}, \\ \langle E - \mu \mathcal{Q}; E - \mu \mathcal{Q} \rangle_{k,\omega} / T &= \gamma T \frac{D_2 k^2}{-i\omega + D_2 k^2} + 4\pi^2 \mathcal{E}^2 K T \frac{D_1 k^2}{-i\omega + D_1 k^2}. \end{aligned} \quad (2.49)$$

Now comparing (2.49) with (1.10), and using the susceptibility matrix (2.41), we can work out the diffusion matrix, D , leading to the result presented earlier in Eq. (1.11). Using (1.12), we find the conductivity matrix

$$\begin{pmatrix} \sigma & \alpha \\ \alpha T & \bar{\kappa} \end{pmatrix} = \begin{pmatrix} D_1 K & 2\pi K \mathcal{E} D_1 \\ 2\pi K \mathcal{E} D_1 T & (\gamma D_2 + 4\pi^2 \mathcal{E}^2 K D_1) T \end{pmatrix}. \quad (2.50)$$

From this result we obtain the Seebeck coefficient presented in Eq. (1.13). Also, we have the thermal conductivity

$$\kappa = \bar{\kappa} - \frac{T\alpha^2}{\sigma} = \gamma D_2 T. \quad (2.51)$$

All of these hydrodynamic results are in accord with the linearized equations of motion

$$\begin{pmatrix} \partial \mathcal{Q} / \partial t \\ \partial E / \partial t - \mu \partial \mathcal{Q} / \partial t \end{pmatrix} = D \begin{pmatrix} \nabla^2 \mathcal{Q} \\ \nabla^2 E - \mu \nabla^2 \mathcal{Q} \end{pmatrix}. \quad (2.52)$$

with the diffusion matrix as in Eq. (1.11). The dynamic susceptibilities (2.49) can be diagonalized by using the operators \mathcal{Q} and $E - \mathcal{Q}(\mu + 2\pi\mathcal{E}T)$. The first of these is carried only by the mode with diffusion constant D_1 , and the second is carried only by the mode with diffusion constant D_2 . D_1 is the charge diffusion constant. We call D_2 the thermal diffusion constant as it's very simply related to the thermal conductivity via Eqs. (2.51) and (2.42).

We can also compute the Wiedemann-Franz ratio, L , from the above results and the computations in Appendix H. Eq. (2.48) leads directly to Eq. (1.14).

The Lyapunov exponent λ_L and butterfly velocity v_B , which characterize early-time chaotic growth through the growth of connected out-of-time-ordered thermal four-point functions

$$\langle\langle V^\dagger(t, \vec{x}) W^\dagger(0) V(t, \vec{x}), W(0) \rangle\rangle_\beta \sim e^{\lambda_L(t - |\vec{x}|/v_B)}, \quad (2.53)$$

can be computed in this model just as in Ref. 57. The key observation here is that these properties are associated with the fluctuations of the ϵ mode, and the ϕ mode is mostly a spectator. Taking V and W to be the SYK fermions, the present model has a Lyapunov exponent given by

$$\lambda_L = 2\pi T, \quad (2.54)$$

saturating the chaos bound of [74]. As in the works [57, 75–77], we find that the butterfly velocity is simply related to the thermal diffusivity as

$$D_2 = \frac{v_B^2}{2\pi T}. \quad (2.55)$$

From Eq. (2.48), we observe that the relationship between v_B and the charge diffusion constant D_1 is not universal [78]: it depends upon the specific parameters of the SYK model.

III. HOLOGRAPHIC THEORIES

The presence, or lack, of translational symmetry has a qualitative impact on the IR transport properties of a system. The disorder present in the higher-dimensional SYK models breaks translational symmetry, resulting in finite thermoelectric conductivities and diffusive transport.

In this Section we study thermoelectric transport in holographic models which do not have translational symmetry. The simplest holographic theories with this property are “bottom-up” models in which translational symmetry is broken isotropically and homogeneously by massless scalar fields φ_i [24–27], with an action [24, 28],

$$S = \int d^4x \sqrt{-g} \left(\mathcal{R} + V - \frac{1}{2} \sum_{i=1}^2 (\partial\varphi_i)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right), \quad (3.1)$$

where $V = 6$. The index i labels the spatial directions of the dual field theory. We will consider more general theories which have AdS_2 near-horizon solutions in Appendix I. This theory has charged black brane solutions

$$\begin{aligned} ds^2 &= -r^2 f(r) dt^2 + \frac{dr^2}{r^2 f(r)} + r^2 d\vec{x}^2, \quad \varphi_i = m x^i, \quad A_t(r) = \mu \left(1 - \frac{r_0}{r} \right), \\ f(r) &= 1 - \frac{m^2}{2r^2} - \left(1 - \frac{m^2}{2r_0^2} + \frac{\mu^2}{4r_0^2} \right) \frac{r_0^3}{r^3} + \frac{\mu^2 r_0^2}{4r^4}. \end{aligned} \quad (3.2)$$

The advantage of breaking translational symmetry in this way is that the black brane solution (3.2), in particular the metric, is very simple.

The bottom-up model (3.1) has not yet been embedded into string theory, and we do not know if it’s possible to do so.³ However, as we are interested in transport properties which we expect to be robust for all systems with the same low-energy symmetries, we forge ahead and determine the properties of the putative field theory dual using the usual AdS/CFT dictionary. This would-be field theory is a (2+1)-dimensional CFT, deformed by a temperature T , a chemical potential μ for a conserved $\text{U}(1)$ charge, and sources $m x^i$ for scalar operators. It is these sources, linear in the spatial coordinates, which explicitly break translational symmetry in the field theory. These sources are not spatially disordered, but they can be thought of as capturing the homogeneous $k = 0$ mode of a disorder sum, which is the most relevant one at small m [13].

The sources μ and m alter the solution, and the result is a geometry that interpolates between AdS_4 near the boundary $r \rightarrow \infty$, and $\text{AdS}_2 \times \mathbb{R}^2$ near the horizon $r = r_0$. To see the near-horizon AdS_2 explicitly, it is convenient to change variables from m to r_* , the location of the zero temperature horizon,

$$12r_*^4 = (2m^2 r_*^2 + \mu^2 r_0^2). \quad (3.3)$$

We then introduce a formal expansion parameter ϵ and make the change of variables

$$\zeta = \frac{r - r_*}{\epsilon}, \quad \zeta_0 = \frac{r_0 - r_*}{\epsilon}, \quad \tau = \epsilon t, \quad (3.4)$$

³ If it were possible, then the dual field theory would be a CFT with two marginal deformations and a flat conformal manifold.

and take the $\epsilon \rightarrow 0$ limit. This limit gives the geometry near the horizon, at small temperatures. The result is

$$\begin{aligned} ds^2 &= -\frac{\zeta^2 f(\zeta)}{\tilde{L}^2} d\tau^2 + \frac{\tilde{L}^2}{\zeta^2 f(\zeta)} d\zeta^2 + d\vec{x}^2 r_*^2 + O(\epsilon), & f(\zeta) &= 1 - \frac{\zeta_0^2}{\zeta^2}, \\ A_\tau &= \frac{\mathcal{E}}{\tilde{L}^2} (\zeta - \zeta_0) + O(\epsilon), & \mathcal{E} &= \frac{\mu}{r_* \left(3 + \frac{\mu^2}{4r_*^2}\right)}, & \varphi_i &= m x^i. \end{aligned} \quad (3.5)$$

At leading order in ϵ , the solution is $\text{AdS}_2 \times \mathbb{R}^2$ with a non-zero electric field \mathcal{E} and scalars φ_i . The AdS_2 radius of curvature is

$$\tilde{L}^2 = \frac{1}{3 + \frac{\mu^2}{4r_*^2}}. \quad (3.6)$$

We are working with units in which the radius of the asymptotically AdS_4 spacetime is unity. This near-horizon geometry is supported by both the gauge field and the scalars, and survives in either of the limits $m \rightarrow 0$ or $\mu \rightarrow 0$.

As was outlined in the introduction, there is a very close connection between the low-energy physics of the SYK models and gravity in nearly AdS_2 spacetimes, as both are governed by the same symmetry-breaking pattern. The presence of the AdS_2 factor in this near-horizon geometry suggests that the low energy physics of the model (3.1) may coincide with that of the SYK model. This is what we will explore and make more precise in the following Subsections.

A. Thermodynamics

The thermodynamic properties of the solution (3.2) can be determined in the usual way [24]. After an appropriate holographic renormalization, evaluating the on-shell Euclidean action gives the grand potential density

$$\Omega(\mu, T) = -\frac{1}{24} \left(4\pi T + \sqrt{6m^2 + 16\pi^2 T^2 + 3\mu^2} \right) \left[2m^2 + \mu^2 + \frac{1}{9} \left(4\pi T + \sqrt{6m^2 + 16\pi^2 T^2 + 3\mu^2} \right)^2 \right], \quad (3.7)$$

where the temperature is calculated from regularity of the Euclidean solution at the horizon

$$T = \frac{r_0^2 f'(r_0)}{4\pi} = \frac{r_0}{4\pi} \left(3 - \frac{m^2}{2r_0^2} - \frac{\mu^2}{4r_0^2} \right), \quad (3.8)$$

and the chemical potential is determined by the value of the gauge field at the AdS_4 boundary

$$\mu \equiv \lim_{r \rightarrow \infty} A_t(r). \quad (3.9)$$

The entropy density \mathcal{S} and charge density \mathcal{Q} are given by the usual thermodynamic derivatives

$$\begin{aligned}\mathcal{S} &\equiv - \left(\frac{\partial \Omega}{\partial T} \right)_\mu = \frac{\pi}{9} \left(3(2m^2 + \mu^2) + 8\pi T \left(4\pi T + \sqrt{6m^2 + 16\pi^2 T^2 + 3\mu^2} \right) \right) = 4\pi r_0^2, \\ \mathcal{Q} &\equiv - \left(\frac{\partial \Omega}{\partial \mu} \right)_T = \frac{\mu}{6} \left(4\pi T + \sqrt{6m^2 + 16\pi^2 T^2 + 3\mu^2} \right) = \mu r_0,\end{aligned}\tag{3.10}$$

These expressions agree with those obtained by using the Bekenstein-Hawking formula to obtain \mathcal{S} in terms of the area of the horizon, and with identifying \mathcal{Q} with the radially conserved electric flux $r^2 A'_t(r)$. Finally, the energy density is

$$\begin{aligned}E &= \Omega + T\mathcal{S} + \mu\mathcal{Q} = 2r_0^3 \left(1 - \frac{m^2}{2r_0^2} + \frac{\mu^2}{4r_0^2} \right) \\ &= \frac{1}{27} \left(4\pi T + \sqrt{6m^2 + 16\pi^2 T^2 + 3\mu^2} \right) \left(-3m^2 + 8\pi^2 T^2 + 3\mu^2 + 2\pi T \sqrt{6m^2 + 16\pi^2 T^2 + 3\mu^2} \right).\end{aligned}\tag{3.11}$$

From these expressions, it is straightforward to compute the susceptibility matrix χ_s . In the low-temperature limit, it is

$$\chi_s = \begin{pmatrix} \frac{m^2 + \mu^2}{\sqrt{3(2m^2 + \mu^2)}} & \frac{2\pi\mu}{3} \\ \frac{2\pi T\mu}{3} & \frac{8\pi^2 T \sqrt{2m^2 + \mu^2}}{3\sqrt{3}} \end{pmatrix}.\tag{3.12}$$

In the notation of (2.42), the $T \rightarrow 0$ limit of the charge susceptibility K and the specific heat at fixed charge γ are

$$K = \frac{m^2 + \mu^2}{\sqrt{3(2m^2 + \mu^2)}}, \quad \gamma = \frac{4\pi^2(2m^2 + \mu^2)^{3/2}}{3\sqrt{3}(m^2 + \mu^2)}.\tag{3.13}$$

Finally, it is straightforward to verify that the relation (1.7) is true in the $T \rightarrow 0$ limit,

$$\lim_{T \rightarrow 0} \left(\frac{\partial \mathcal{S}}{\partial \mathcal{Q}} \right)_T = 2\pi\mathcal{E} = \frac{2\pi\mu}{\sqrt{3}} \frac{\sqrt{2m^2 + \mu^2}}{m^2 + \mu^2}.\tag{3.14}$$

As we have emphasized, the UV physics of this holographic theory is quite different from that of the SYK models. We expect there to be similarities in their IR properties due to the near-horizon AdS_2 geometry. It is then important to determine which of the thermodynamic properties we have just described are universal, *i.e.* are determined solely by the near-horizon geometry, and which are not and depend upon the UV details of the solution. A naive guess would be that the AdS_2 geometry captures the $T \rightarrow 0$ limit of the full thermodynamics, but this is not quite right.

As in our analysis of the SYK models, it is much more convenient to work with the canonical free energy density $F(\mathcal{Q}, T) = \Omega + \mu\mathcal{Q}$. In the low-temperature limit, this has the form of (1.5)

$$F(\mathcal{Q}, T) = E_0(\mathcal{Q}) - T\mathcal{S}_0(\mathcal{Q}) + O(T^2),\tag{3.15}$$

where E_0 and \mathcal{S}_0 are the $T = 0$ limits of the energy (3.11) and entropy (3.10) densities at fixed \mathcal{Q}

$$\begin{aligned} E_0(\mathcal{Q}) &= \mathcal{Q} \sqrt{\sqrt{m^4 + 12\mathcal{Q}^2} - m^2} - \frac{1}{6\sqrt{3}} \left(\sqrt{m^4 + 12\mathcal{Q}^2} + m^2 \right)^{3/2}, \\ \mathcal{S}_0(\mathcal{Q}) &= \frac{\pi}{3} \left(\sqrt{m^4 + 12\mathcal{Q}^2} + m^2 \right). \end{aligned} \quad (3.16)$$

This is more convenient because, from the point of view of the near-horizon geometry, \mathcal{Q} is a much more natural object than μ . Of the four thermodynamic objects $T, \mathcal{S}, \mathcal{Q}$ and μ , only the first three can be determined from just the near-horizon solution: T from regularity of the Euclidean near-horizon geometry, \mathcal{S} from the area of the horizon, and – for a solution with trivial profiles for charged matter fields – \mathcal{Q} from the radially conserved electric flux, which can be evaluated at the horizon. In contrast, μ (defined by (3.9)) requires knowledge of the UV part of the geometry. For this reason, we will find that thermodynamic quantities involving μ will in general be non-universal, *i.e.* they depend upon the UV parts of the geometry. The charge susceptibility $(\partial\mathcal{Q}/\partial\mu)_T$ is one such non-universal quantity.

1. Dimensional reduction

To make this more precise, we will cut off our geometry at the boundary of the near-horizon spacetime (3.5), and study the thermodynamics of this solution. So that we may use the usual AdS/CFT dictionary, we will compactify the \mathbb{R}^2 part of the geometry on a torus of volume V_2 , and study the resulting asymptotically AdS₂ solution within a 2D theory of gravity. We compactify using the ansatz⁴

$$ds^2 = \sqrt{\phi_0/\phi} d\tilde{s}^2 + \phi d\tilde{x}^2, \quad A_\mu dx^\mu = \tilde{A}_a dx^a, \quad \varphi_i = m x^i, \quad (3.17)$$

where ϕ_0 is a constant and the fields do not depend on the torus coordinates. Reducing the theory (3.2) on the spatial torus gives the two-dimensional Einstein-Maxwell-dilaton action (up to boundary terms)

$$S_{2D} = V_2 \int \sqrt{-\tilde{g}} \left(\phi \tilde{\mathcal{R}} + V(\phi) - \frac{Z(\phi)}{4} \tilde{F}^2 \right), \quad (3.18)$$

where

$$V(\phi) = 6\sqrt{\phi_0\phi} - \left(6r_*^2 - \frac{\mu^2 r_0^2}{2r_*^2} \right) \sqrt{\phi_0/\phi}, \quad Z(\phi) = \sqrt{\phi^3/\phi_0}. \quad (3.19)$$

⁴ Tildes will denote quantities in the dimensionally reduced theory, and the indices a, b run over the uncompactified directions.

An exact solution of the equations of motion of this action is an AdS_2 geometry with constant dilaton and electric field \mathcal{E} ,

$$\begin{aligned} d\tilde{s}^2 &= -\frac{\zeta^2}{\tilde{L}^2} f(\zeta) dt^2 + \frac{\tilde{L}^2}{\zeta^2 f(\zeta)}, & f(\zeta) &= 1 - \frac{\zeta_0^2}{\zeta^2}, & \tilde{A}_t &= \frac{\mathcal{E}}{\tilde{L}^2} (\zeta - \zeta_0), & \phi_0 &= r_*^2, \\ \mathcal{E} &= \frac{\mu r_0 \tilde{L}^2}{r_*^2}, & \tilde{L}^2 &= \frac{1}{3r_*^2 + \frac{\mu^2 r_0^2}{4r_*^2}}. \end{aligned} \quad (3.20)$$

If we set $r_0 = r_*$, this is the compactified version of the near-horizon solution (3.5).

2. Thermodynamics of AdS_2 solutions

We will now determine the thermodynamics of these AdS_2 spacetimes. The action (3.18), for general $V(\phi)$ and $Z(\phi)$, has charged AdS_2 solutions with constant dilaton $\phi = \phi_0$ provided that

$$V(\phi_0) = \frac{\mathcal{E}^2}{2\tilde{L}^4} Z(\phi_0), \quad \text{and} \quad \frac{2}{\tilde{L}^2} = V'(\phi_0) + \frac{\mathcal{E}^2}{2\tilde{L}^4} Z'(\phi_0). \quad (3.21)$$

where the solutions are written as

$$d\tilde{s}^2 = -\frac{\zeta^2 f(\zeta)}{\tilde{L}^2} dt^2 + \frac{\tilde{L}^2}{\zeta^2 f(\zeta)} d\zeta^2, \quad f(\zeta) = 1 - \frac{\zeta_0^2}{\zeta^2}, \quad \tilde{A}_t = \frac{\mathcal{E}}{\tilde{L}^2} (\zeta - \zeta_0). \quad (3.22)$$

Computing the temperature \tilde{T} using regularity of the Euclidean solution and the charge density $\tilde{\mathcal{Q}}$ from the radially conserved electric flux (where densities are now given by dividing by the torus volume V_2), we find

$$\tilde{T} = \frac{\zeta_0}{2\pi\tilde{L}^2}, \quad \tilde{\mathcal{Q}} = \frac{\mathcal{E}}{\tilde{L}^2} Z(\phi_0). \quad (3.23)$$

To compute the free energy of these solutions, we must supplement the action (3.18) with boundary terms. For general $V(\phi)$ and $Z(\phi)$ these are given by⁵

$$S_{bdy} = V_2 \int dt \sqrt{-\tilde{\gamma}} \left(2\phi \tilde{K} - \tilde{L} V(\phi) - Z(\phi) \tilde{A}_a \tilde{F}^{ab} \tilde{n}_b - \frac{\tilde{L}}{4} Z(\phi) \tilde{F}^2 \right), \quad (3.24)$$

where \tilde{n}^a is the normal vector to the boundary, $\tilde{\gamma}$ is the induced metric on the boundary with \tilde{K} its extrinsic curvature, and \tilde{L} is the AdS_2 radius of curvature. With these counterterms, the Euclidean on-shell action gives the canonical free energy density of the AdS_2 solution as

$$\tilde{F}(\tilde{\mathcal{Q}}, \tilde{T}) = -4\pi\tilde{T}\phi_0. \quad (3.25)$$

⁵ There are typos in the corresponding boundary terms written in [48], which we have corrected here. These boundary terms agree with those in [55] for the particular theory studied there.

The right hand side is a non-trivial function of \mathcal{Q} due to the implicit dependence of ϕ_0 on \mathcal{Q} given by equations (3.21). For small variations, these equations imply that

$$\delta\phi_0 = \frac{\mathcal{E}}{2}\delta\tilde{\mathcal{Q}}. \quad (3.26)$$

Taking variations of the canonical free energy density gives the chemical potential and entropy density [48]

$$\tilde{\mu} = \left(\frac{\partial \tilde{F}}{\partial \tilde{\mathcal{Q}}} \right)_{\tilde{T}} = -2\pi\mathcal{E}\tilde{T}, \quad \text{and} \quad \tilde{\mathcal{S}} = - \left(\frac{\partial \tilde{F}}{\partial \tilde{T}} \right)_{\tilde{\mathcal{Q}}} = 4\pi\phi_0. \quad (3.27)$$

One further application of this formula gives

$$\left(\frac{\partial \tilde{\mathcal{S}}}{\partial \tilde{\mathcal{Q}}} \right)_{\tilde{T}} = - \left(\frac{\partial \tilde{\mu}}{\partial \tilde{T}} \right)_{\tilde{\mathcal{Q}}} = 2\pi\mathcal{E}. \quad (3.28)$$

By comparing with (3.14), we see that the AdS₂ solution of the 2-dimensional theory (3.18) correctly captures the $T = 0$ limit of the thermodynamic function $(\partial\mathcal{S}/\partial\mathcal{Q})_T$ of the full, asymptotically AdS₄ solution. It does not capture the small T corrections.

In fact, for the case (3.19) which arises from dimensional reduction of the complete solution (3.2), we find that the canonical free energy density of the AdS₂ geometry can be written

$$\begin{aligned} \tilde{F}(\tilde{\mathcal{Q}}, \tilde{T}) &= -\tilde{\mathcal{S}}(\tilde{\mathcal{Q}})\tilde{T}, \\ &= -\mathcal{S}_0(\tilde{\mathcal{Q}})\tilde{T}, \end{aligned} \quad (3.29)$$

where $\mathcal{S}_0(\mathcal{Q})$ is (3.16), the zero temperature entropy of the full four dimensional solution. Comparing to (3.15), we see that the linear-in- T part of the free energy density is universal i.e. it is independent of the UV geometry. This is the holographic analogue of the SYK result (1.5).

Comparing (3.15) and (3.29), we see that the T -independent part of the canonical free energy is not universal: it depends upon how the AdS₂ near-horizon geometry is embedded into the full solution. This results in a non-trivial “renormalization” of the chemical potential of the AdS₂ solution $\tilde{\mu}$ with respect to the chemical potential of the full solution μ ,

$$\tilde{\mu}(\mathcal{Q}, T) = \mu(\mathcal{Q}, T) - \mu_0(\mathcal{Q}) + O(T^2), \quad (3.30)$$

where

$$\mu_0(\mathcal{Q}) = \frac{\partial E_0}{\partial \mathcal{Q}} = \sqrt{\sqrt{m^4 + 12\mathcal{Q}^2} - m^2}, \quad (3.31)$$

is the $T = 0$ chemical potential of the full solution, which depends upon the UV geometry. The linear-in- T components of μ and $\tilde{\mu}$ agree because they are related to the universal quantity $(\partial\mathcal{S}/\partial\mathcal{Q})_T$ by the Maxwell relation (1.7).

One result of this renormalization of μ is that the low T limit of the charge susceptibility of the full solution, K in equation (3.13), is unrelated to the charge susceptibility of the AdS_2 solution of the two-dimensional action (3.18). Explicitly, the charge susceptibility of the AdS_2 solution is

$$\left(\frac{\partial \tilde{\mathcal{Q}}}{\partial \tilde{\mu}}\right)_{\tilde{T}} = -\frac{1}{2\pi\tilde{T}} \frac{Z(\phi_0)}{\tilde{L}^2} \left(\left(1 - \frac{\mathcal{E}^2 Z'(\phi_0)}{2\tilde{L}^2}\right)^2 - \frac{Z(\phi_0)\mathcal{E}^2}{4} V''(\phi_0) - \frac{Z(\phi_0)\mathcal{E}^4}{8\tilde{L}^4} Z''(\phi_0) \right)^{-1}, \quad (3.32)$$

which diverges as $1/T$, unlike K . Thus, the $T = 0$ charge susceptibility cannot be obtained from an effective two dimensional action for the near-horizon part of the geometry.

Finally, let us address the $O(T^2)$ terms in the free energy (3.15), which are responsible for the $T = 0$ limit of the specific heat at constant \mathcal{Q} of the full solution, γ in equation (3.13). The two-dimensional theory does not have terms like this, and therefore has a vanishing specific heat. For the uncharged case, Ref. 48–50 showed that the leading contribution to the specific heat can be found by including the correction to the dilaton which grows towards the AdS_2 boundary. In principle, the inclusion of corrections like this should also lead to a non-vanishing specific heat in the charged case, but this is beyond the scope of this paper.

B. Transport

The transport properties of the field theory dual to the solution (3.2) have been studied in great detail in [21–23, 29–31, 76, 79–81]. When translational symmetry is broken, which in this case means $m \neq 0$, transport of heat and charge over the longest distances and timescales should be governed by the equations (1.8) and (1.10) of diffusive hydrodynamics. For the $\mathcal{Q} = 0$ case, this was checked numerically in [29]. Given that the susceptibility matrix is (3.12), the remaining quantities characterizing the transport of the system are the three elements of the dc conductivity matrix (1.12). With this information, the diffusion matrix (1.12) and response functions (1.10) are fixed by the theory of diffusive hydrodynamics.

It is not unreasonable to expect a connection between the dc conductivities of the higher-dimensional SYK theory, and those of the holographic theory (3.2), because in the holographic case these are determined by the AdS_2 horizon. In general, for a given UV gravitational theory with asymptotically AdS solutions and without translational symmetry, the dc conductivities are given by properties of the gravitational solution at the horizon [16, 23, 80, 82, 83]. One does not need to know how the near-horizon solutions (which may or may not be AdS_2) are embedded into the full solution.

For the solution (3.2), the dc conductivities are

$$\sigma = 1 + \frac{\mu^2}{m^2}, \quad \alpha = \frac{4\pi\mathcal{Q}}{m^2}, \quad \bar{\kappa} = \frac{4\pi\mathcal{S}T}{m^2}. \quad (3.33)$$

In the low T limit, the Seebeck coefficient $S \equiv \frac{\alpha}{\sigma}$ is

$$\lim_{T \rightarrow 0} S = \frac{2\pi\mu\sqrt{2m^2 + \mu^2}}{\sqrt{3}(m^2 + \mu^2)} = \lim_{T \rightarrow 0} \left(\frac{\partial \mathcal{S}}{\partial \mathcal{Q}} \right)_T = 2\pi\mathcal{E}, \quad (3.34)$$

as advertised in (1.13). This is a non-trivial relation between three quantities associated to the AdS_2 near-horizon geometry. Although the final equality can be derived from the simple two-dimensional action (3.18), this action alone is not sufficient for determining the dc conductivities, which depend upon the correlation functions of spatial currents, or equivalently upon the correlation functions of gradients of the charge and energy densities.

Due to the relation (3.34), the low T response functions of the holographic theory take the same form as those of the SYK model (2.49). The low T diffusion constants of the holographic theory are

$$D_1 = \frac{\sqrt{3(2m^2 + \mu^2)}}{m^2}, \quad D_2 = \frac{\sqrt{3}}{\sqrt{2m^2 + \mu^2}}. \quad (3.35)$$

and the charge susceptibility K , specific heat γ , and \mathcal{E} are given in equations (3.13) and (3.5). The functional form of the diffusion constants and thermodynamic functions are different in this holographic model than in the SYK model of Section II E, but the structure of the response functions is the same.

The divergence of one of the diffusion constants in the translationally invariant case $m = 0$ is a consequence of the fact that diffusive hydrodynamics is not applicable in this limit. One of the diffusive excitations is replaced by a propagating sound wave [30]. The diffusive mode which survives in this limit corresponds to diffusion of a certain linear combination of the charge and heat currents [84, 85]. When $m = 0$, the applicability of the relation (3.34) is more subtle. In this case, the dc conductivities α and σ are infinite due to translational invariance. By studying the optical conductivities, one can cleanly distinguish between an infinite and finite contribution to each dc conductivity. The ratio of the infinite contributions is $\alpha_{inf}/\sigma_{inf} = \mathcal{S}/\mathcal{Q} = 2\pi\mathcal{E}$ and so obeys equation (3.34). The ratio of the finite contributions is $\alpha_f/\sigma_f = -\mu/T$. Note that the right-hand-side is given by the full chemical potential, and thus, in this case, this ratio is not a universal quantity. The $m = 0$ theory is special because in this case the conductivities are not simply properties of the AdS_2 horizon [85]. In fact, when $m = 0$, the ratio between the finite contributions is fixed by the UV relativistic symmetry [86].

It is simple to obtain the Wiedemann-Franz ratio L , defined in Eq. (1.14), which is given by

$$\lim_{T \rightarrow 0} L = \frac{4\pi^2 m^2 (2m^2 + \mu^2)}{3(m^2 + \mu^2)^2}, \quad (3.36)$$

at zero temperature. Curiously, the prefactor of $4\pi^2/3$ is the same as that in the SYK model result in Eq. (1.14). Eq. (3.36) vanishes in the translationally invariant limit $m \rightarrow 0$, as expected from

the general arguments of [87]. We can also define the Wiedemann-Franz-like ratio

$$L_\alpha \equiv \frac{\kappa}{\alpha T}, \quad (3.37)$$

where we have replaced the electrical conductivity in the usual Wiedemann-Franz ratio with the thermoelectric conductivity. For this holographic theory, the low temperature limit of L_α is given by a simple thermodynamic formula

$$\lim_{T \rightarrow 0} L_\alpha = -\mathcal{Q} \frac{\partial}{\partial \mathcal{Q}} \left(\frac{\mathcal{S}_0}{\mathcal{Q}} \right) = \frac{2\pi m^2 \sqrt{2m^2 + \mu^2}}{\sqrt{3}\mu (m^2 + \mu^2)}. \quad (3.38)$$

The first equality is a consequence of both the ‘Kelvin formula’ (1.13), and the relation $\bar{\kappa}/\alpha T = \mathcal{S}/\mathcal{Q}$. This latter relation is a generic property of holographic theories with homogeneous translational symmetry breaking, and is unrelated to the existence of an AdS_2 near-horizon geometry. It is therefore unsurprising that this property, and hence the thermodynamic relation in (3.38), are not shared by the SYK models. But the thermodynamic formula for L_α does extend to more general holographic theories with AdS_2 horizons (see appendix I).

IV. CONCLUSIONS

This paper has presented the thermodynamic and transport properties of two classes of solvable models of diffusive metallic states without quasiparticle excitations. Both classes of models conserve total energy and a $U(1)$ charge, \mathcal{Q} , but do not conserve total momentum. The first class concerns the higher-dimensional SYK models of fermions with random $q/2$ -body interactions. The second class involves a holographic mapping to gravitational theories of black branes with an AdS_2 near-horizon geometry. We found that these classes shared a number of common properties:

- The low T thermodynamics is described by the free energy in Eq. (1.5), with the entropy $\mathcal{S}(\mathcal{Q})$ universal, and the ground state energy $E_0(\mathcal{Q})$ non-universal. For the SYK models, universality implies dependence only on the IR scaling dimension of the fermion, and independence from possible higher-order interactions in the Hamiltonian. In the holography, universality implies independence from the geometry far from the AdS_2 near-horizon geometry.
- The thermoelectric transport is constrained by a simple expression (Eq. (1.13)) equating the Seebeck coefficient to the \mathcal{Q} -derivative of the entropy \mathcal{S} . This is the ‘Kelvin formula’ proposed in Ref. 68 by different approximate physical arguments. In our analysis, the Kelvin formula was the consequence of an emergent $\text{PSL}(2, \mathbb{R})$ symmetry shared by both classes of models.

- As has also been discussed earlier [40], the correlators of non-conserved local operators have a form (see Eq. (2.9)) constrained by conformal invariance, and characterized by a spectral asymmetry parameter, \mathcal{E} , which is defined by Eq. (1.7); see also Appendix B. In the holographic context, \mathcal{E} also has the interpretation as the strength of an electric field in AdS_2 .
- Both classes of models [36, 47, 88] saturate the bound [74] on the Lyapunov rate which characterizes the growth of quantum chaos, $\lambda_L = 2\pi T$.
- For the SYK models, the butterfly velocity, v_B , was found to be universally related to the thermal diffusivity, D_2 by Eq. (2.55), as in Ref. 57. On the other hand, the SYK models do not display a universal relation between v_B and the charge diffusivity, D_1 . So the universal connection between chaos and transport is restricted to energy transport, as was also found in the study of a critical Fermi surface [77]. Chaos is naturally connected to energy fluctuations, because the local energy determines the rate of change of the phase of the quantum state, and phase decoherence is responsible for chaos. This physical argument finds a direct realization in the computation on the SYK model. In the holographic axion model with $\mu = 0$, the relationship between $D_{1,2}$ and v_B was investigated in Ref. 76, and D_2 was found to obey Eq. (2.55).

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Appendix A: Saddle point solution of the SYK model

We follow the condensed matter notation for Green’s functions in which

$$G(\tau) = -\langle T_\tau(f(\tau)f^\dagger(0)) \rangle. \quad (\text{A1})$$

It is useful to make ansatzes for the retarded Green's functions in the complex frequency plane, because then the constraints from the positivity of the spectral weight are clear. At the Matsubara frequencies, the Green's function is defined by

$$G(i\omega_n) = \int_0^{1/T} d\tau e^{i\omega_n \tau} G(\tau). \quad (\text{A2})$$

So the bare Green's function is

$$G_0(i\omega_n) = \frac{1}{i\omega_n + \mu}. \quad (\text{A3})$$

The Green's functions are continued to all complex frequencies z via the spectral representation

$$G(z) = \int_{-\infty}^{\infty} \frac{d\Omega}{\pi} \frac{\rho(\Omega)}{z - \Omega}. \quad (\text{A4})$$

For fermions, the spectral density obeys

$$\rho(\Omega) > 0, \quad (\text{A5})$$

for all real Ω and T . The retarded Green's function is $G^R(\omega) = G(\omega + i\eta)$ with η a positive infinitesimal, while the advanced Green's function is $G^A(\omega) = G(\omega - i\eta)$. It is also useful to tabulate the inverse Fourier transforms at $T = 0$

$$G(\tau) = \begin{cases} -\int_0^{\infty} \frac{d\Omega}{\pi} \rho(\Omega) e^{-\Omega\tau} & , \text{ for } \tau > 0 \text{ and } T = 0 \\ \int_0^{\infty} \frac{d\Omega}{\pi} \rho(-\Omega) e^{\Omega\tau} & , \text{ for } \tau < 0 \text{ and } T = 0 \end{cases}. \quad (\text{A6})$$

Using (A6) we obtain in τ space

$$G(\tau) = \begin{cases} -\frac{C\Gamma(2\Delta) \sin(\pi\Delta + \theta)}{\pi|\tau|^{2\Delta}} & , \text{ for } \tau > 0 \text{ and } T = 0 \\ \frac{C\Gamma(2\Delta) \sin(\pi\Delta - \theta)}{\pi|\tau|^{2\Delta}} & , \text{ for } \tau < 0 \text{ and } T = 0 \end{cases}. \quad (\text{A7})$$

We also use the spectral representations for the self energies

$$\Sigma(z) = \int_{-\infty}^{\infty} \frac{d\Omega}{\pi} \frac{\sigma(\Omega)}{z - \Omega}. \quad (\text{A8})$$

Using (A7) and (2.1) we obtain

$$\sigma(\Omega) = \begin{cases} \frac{\pi J^2 q}{\Gamma(2(q-1)\Delta)} \left[\frac{C\Gamma(2\Delta)}{\pi} \right]^{q-1} [\sin(\pi\Delta + \theta)]^{q/2} [\sin(\pi\Delta - \theta)]^{q/2-1} |\Omega|^{2\Delta(q-1)-1} & , \text{ for } \Omega > 0 \\ \frac{\pi J^2 q}{\Gamma(2(q-1)\Delta)} \left[\frac{C\Gamma(2\Delta)}{\pi} \right]^{q-1} [\sin(\pi\Delta + \theta)]^{q/2-1} [\sin(\pi\Delta - \theta)]^{q/2} |\Omega|^{2\Delta(q-1)-1} & , \text{ for } \Omega < 0 \end{cases} \quad (\text{A9})$$

Now from (2.2) we have in the IR limit

$$\Sigma(z) = -\frac{1}{C} e^{i(\pi\Delta + \theta)} z^{(1-2\Delta)}. \quad (\text{A10})$$

So comparing (A9) and (A10) we have the solutions in Eqs. (1.4) and (2.5).

Appendix B: Constraints from conformal invariance at nonzero μ

In Eq. (2.3), we made an ansatz for the form of the low-frequency two-point function of the SYK fermion at nonzero chemical potential. In this Appendix we show that this ansatz follows from the assumption of a low-energy conformal invariance, which unlike in higher-dimensional quantum field theory, can arise in zero or one spatial dimensions.

To see this, it is helpful to imagine coupling a $(0+1)$ -dimensional quantum theory with a $U(1)$ global symmetry to an external metric and external $U(1)$ gauge field. Suppose the theory is on the Euclidean line and that the external gauge field corresponds to a chemical potential $\tilde{\mu}$, $A_\tau = -i\tilde{\mu}$. When $\tilde{\mu} = 0$, this background is invariant under global conformal transformations,

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad ad - bc = 1. \quad (\text{B1})$$

The group of global conformal transformations is isomorphic to $\text{PSL}(2, \mathbb{R})$. When $\tilde{\mu} \neq 0$, the coordinate transformation (B1) does not leave the external gauge field invariant, but the combination of (B1) and a gauge transformation

$$\Lambda = i\tilde{\mu} \left(\frac{a\tau + b}{c\tau + d} - \tau \right), \quad (\text{B2})$$

does, under the convention that A_τ transforms under gauge transformations as $A_\tau \rightarrow A_\tau + \partial_\tau \Lambda$. So a $\text{PSL}(2, \mathbb{R})$ global conformal symmetry may be maintained even at nonzero chemical potential.

This global conformal group is generated by a time translation H , dilatation D , and a special conformal transformation K . As we usually do, let a primary operator be one which is annihilated by K . Primary operators are labeled by their dimension Δ and $U(1)$ charge, which we henceforth take to be unity. Using that a conformal transformation is the combination of a coordinate transformation (B1) and gauge transformation (B2), the action of an infinitesimal conformal transformation $\delta\tau = f$ and an independent, infinitesimal gauge transformation λ on a primary operator \mathcal{O} is given by

$$\delta_f \mathcal{O} = -f(\partial_\tau - iA_\tau)\mathcal{O} - \Delta(\partial_\tau f)\mathcal{O} + i\lambda\mathcal{O}. \quad (\text{B3})$$

Observe that, after Fourier transforming τ to a Euclidean frequency ω_E , the action of the conformal transformations at $\tilde{\mu} \neq 0$ is the same as at $\tilde{\mu} = 0$, but with the substitution $\omega_E \rightarrow \omega_E - i\tilde{\mu}$. Thus, up to a change in the normalization, the frequency-space two-point function of \mathcal{O} at nonzero $\tilde{\mu}$ is just given by the two-point function at $\tilde{\mu} = 0$ but with this same replacement.

At zero temperature this just gives that the two-point function of \mathcal{O} is proportional to $(\omega_E - i\tilde{\mu})^{2\Delta-1}$, which recovers the ansatz (2.3) with $z^{-1} = \omega_E - i\tilde{\mu}$. At nonzero temperature $T = 1/\beta$, a similar, but lengthier argument shows that the two-point function of \mathcal{O} is given by

$$G(i\omega_n) = -\frac{iC e^{-i\theta}}{\beta^{2\Delta-1}} \frac{\Gamma\left(\Delta - \frac{\beta}{2\pi}(\omega_n - i\tilde{\mu})\right)}{\Gamma\left(1 - \Delta - \frac{\beta}{2\pi}(\omega_n - i\tilde{\mu})\right)}, \quad (\text{B4})$$

where c is a constant and θ is the same phase appearing in (2.3). This phase is related to $\tilde{\mu}$ and T in the following way. Fourier transforming back to Euclidean time τ , the Euclidean Green's function must be a real function of τ . Using that the Matsubara frequencies for fermions are $\omega_n = 2\pi(n + \frac{1}{2})$, we find after some algebra that for fermionic \mathcal{O} , θ and μ are related as

$$e^{-\beta\tilde{\mu}} = \frac{\sin(\pi\Delta + \theta)}{\sin(\pi\Delta - \theta)}. \quad (\text{B5})$$

This coincides with the expression (2.7) relating θ and \mathcal{E} , provided that we identify

$$2\pi\mathcal{E} = -\frac{\tilde{\mu}}{T}. \quad (\text{B6})$$

For now, take this expression to define the spectral asymmetry \mathcal{E} . We conclude this Appendix by arguing that this definition of \mathcal{E} also satisfies (2.8).

Scale invariance implies that the canonical ensemble free energy has the form

$$F(\mathcal{Q}, T) = -T\mathcal{S}(\mathcal{Q}), \quad (\text{B7})$$

where $\mathcal{S}(\mathcal{Q})$ is the zero-temperature entropy. The chemical potential $\tilde{\mu}$ is then

$$\tilde{\mu}(\mathcal{Q}, T) = -T \frac{d\mathcal{S}}{d\mathcal{Q}}. \quad (\text{B8})$$

Eq. (B6) trivially implies

$$2\pi\mathcal{E} = -\lim_{T \rightarrow 0} \frac{\partial^2 F}{\partial T \partial \mathcal{Q}} = -\lim_{T \rightarrow 0} \left(\frac{\partial \tilde{\mu}}{\partial T} \right)_{\mathcal{Q}}, \quad (\text{B9})$$

which is what we wanted to show.

Appendix C: Large q expansion of the SYK model

Section II C obtained exact results for the universal parts of the thermodynamic observables. However, no explicit results for the non-universal parts dependent upon J . In this appendix we will present the large q expansion of the Hamiltonian in Eq. (1.1): the results contain both the universal and non-universal parts.

We begin by recalling the universal results of Section II C in the limit of small $\Delta = 1/q$. At low T , the thermodynamics contains 3 universal quantities which do not undergo any UV renormalization: they are the density, \mathcal{Q} , the entropy \mathcal{S} , and the ‘electric field’ \mathcal{E} . All 3 quantities can be expressed in terms of universal expressions of each other. First, we treat \mathcal{Q} as the independent quantity. Then, the $T \rightarrow 0$ limit of the entropy is from (2.13), (2.20), (2.21),

$$\mathcal{S}(\mathcal{Q}) = \mathcal{Q} \ln \left(\frac{1 - 2\mathcal{Q}}{1 + 2\mathcal{Q}} \right) + \frac{1}{2} \ln \left(\frac{4}{1 - 4\mathcal{Q}^2} \right) - \frac{\pi^2}{2} (1 - 4\mathcal{Q}^2) \Delta^2 + \mathcal{O}(\Delta^3). \quad (\text{C1})$$

By taking a \mathcal{Q} derivative, we have immediately

$$\mathcal{E}(\mathcal{Q}) = \frac{1}{2\pi} \ln \left(\frac{1-2\mathcal{Q}}{1+2\mathcal{Q}} \right) + 2\pi\mathcal{Q}\Delta^2 + \mathcal{O}(\Delta^3). \quad (\text{C2})$$

Next, we take \mathcal{E} as the independent variable. Then the inverse function (C2) is

$$\mathcal{Q}(\mathcal{E}) = -\frac{1}{2} \tanh(\pi\mathcal{E}) - \frac{\pi^2 \sinh(\pi\mathcal{E})}{2 \cosh^3(\pi\mathcal{E})} \Delta^2 + \mathcal{O}(\Delta^3). \quad (\text{C3})$$

The entropy is given by (2.21), $\mathcal{S}(\mathcal{E}) = \mathcal{G}(\mathcal{E}) + 2\pi\mathcal{E}\mathcal{Q}(\mathcal{E})$, where

$$\mathcal{G}(\mathcal{E}) = \ln(2 \cosh(\pi\mathcal{E})) - \frac{\pi^2}{2 \cosh^2(\pi\mathcal{E})} \Delta^2 + \mathcal{O}(\Delta^3). \quad (\text{C4})$$

Now we turn to the explicit large q expansion to compute the thermodynamics in terms of microscopic parameters. The expressions here depend upon the underlying J , and the specific form of the Hamiltonian in Eq. (1.1). We will verify that they are compatible with the universal results presented above.

The large q expansion was presented by Ref. 47 at $\mu = 0$, and we follow their analysis here. At $q = \infty$ they showed that the Green's function was that of a dispersionless free fermion. So, we write

$$G_s(\tau) = G_0(\tau) \left[1 + \frac{1}{q} G_1(\tau) \right] \quad (\text{C5})$$

where the dispersionless free fermion Green's function is

$$G_0(\tau) = \begin{cases} -e^{\mu\tau}(e^{\mu/T} + 1)^{-1}, & , \quad 0 < \tau < 1/T \\ e^{\mu\tau}(e^{-\mu/T} + 1)^{-1}, & , \quad -1/T < \tau < 0 \end{cases}. \quad (\text{C6})$$

Then from (2.1) we have the self energy

$$\Sigma_s(\tau) = -\frac{qJ^2 e^{\mu\tau}}{(e^{\mu/T} + 1)(2 + 2 \cosh(\mu/T))^{q/2-1}} \left[1 + \frac{1}{q} G_1(\tau) \right]^{q/2} \left[1 + \frac{1}{q} G_1(-\tau) \right]^{q/2-1} \quad (\text{C7})$$

Now we define

$$\mathcal{J}^2 = \frac{q^2 J^2}{2(2 + 2 \cosh(\mu/T))^{q/2-1}}. \quad (\text{C8})$$

The large q can only be taken if we adjust the bare J so that \mathcal{J} is q independent. To the order we shall work, it is legitimate to use the $\mathcal{O}(\Delta^0)$ result above, in which case we will find

$$\mathcal{J}^2 = \frac{q^2 J^2}{2(2 + 2 \cosh(2\pi\mathcal{E}))^{q/2-1}}. \quad (\text{C9})$$

As \mathcal{E} is only a function of \mathcal{Q} , we find that \mathcal{J} remains finite as $T \rightarrow 0$. Then, in the large q limit

$$\Sigma_s(\tau) = -\frac{2\mathcal{J}^2}{q} G_0(\tau) \exp \left(\frac{1}{2} (G_1(\tau) + G_1(-\tau)) \right) \quad (\text{C10})$$

In this form, the explicit μ dependence has disappeared. Ref. 47 obtained a differential equation for G_1 at $\mu = 0$, and so this applies also here; the solution is

$$G_1(\tau) = \ln \left(\frac{\cos^2(\pi v/2)}{\cos^2(\pi v(T|\tau| - 1/2))} \right), \quad (\text{C11})$$

where v is obtained by the solution of

$$\frac{\pi v}{\cos(\pi v/2)} = \frac{\mathcal{J}}{T}. \quad (\text{C12})$$

Assuming a fixed \mathcal{J} , the low T expansion of v is

$$v = 1 - \frac{2T}{\mathcal{J}} + \frac{4T^2}{\mathcal{J}^2} + \dots \quad (\text{C13})$$

To compute the grand potential, Ω , we use the effective action

$$\begin{aligned} S[G, \Sigma] = & -\text{Tr} \ln \left[\delta(\tau - \tau') \left(-\frac{\partial}{\partial \tau} + \mu \right) - \Sigma(\tau, \tau') \right] + \frac{\mu}{2T} \\ & - \int_0^{1/T} d\tau d\tau' \left[\Sigma(\tau, \tau') G(\tau', \tau) + (-1)^{q/2} J^2 [G(\tau, \tau')]^{q/2} [G(\tau', \tau)]^{q/2} \right]. \end{aligned} \quad (\text{C14})$$

The $G_s(\tau - \tau')$ and $\Sigma_s(\tau - \tau')$ above are the solutions to the saddle-point equations of S . It is simpler to evaluate $d\Omega/d\mathcal{J}$ because only the last term contributes

$$\begin{aligned} \mathcal{J} \frac{d\Omega}{d\mathcal{J}} = & -\frac{4\mathcal{J}^2}{q^2(2 + 2 \cosh(\mu/T))} \int_0^{1/T} d\tau \exp(G_1(\tau)) \\ = & -\frac{4\mathcal{J}^2 \sin(\pi v)}{q^2 \pi T v (2 + 2 \cosh(\mu/T))}, \end{aligned} \quad (\text{C15})$$

which implies

$$\frac{d\Omega}{dv} = -\frac{8\pi T}{q^2(2 + 2 \cosh(\mu/T))} \tan\left(\frac{\pi v}{2}\right) \left[1 + \frac{\pi v}{2} \tan\left(\frac{\pi v}{2}\right) \right] \quad (\text{C16})$$

Integrating over v , we obtain the grand potential as a function of the bare μ and T

$$\Omega(\mu, T) = -T \ln(2 \cosh(\mu/(2T))) - \frac{2\pi v T}{\cosh^2(\mu/(2T))} \left[\tan\left(\frac{\pi v}{2}\right) - \frac{\pi v}{4} \right] \Delta^2 + \mathcal{O}(\Delta^3). \quad (\text{C17})$$

This is the main result of the large q expansion.

Now we can use thermodynamic relations to determine both universal and non-universal observables. From the grand potential in (C17), we have the density

$$\mathcal{Q} = \frac{1}{2} \tanh(\mu/(2T)) - \frac{2\pi v \sinh(\mu/(2T))}{\cosh^3(\mu/(2T))} \left[\tan\left(\frac{\pi v}{2}\right) - \frac{\pi v}{4} \right] \Delta^2 + \mathcal{O}(\Delta^3). \quad (\text{C18})$$

Combining (C17) and (C18), we can obtain the free energy in the canonical ensemble

$$\begin{aligned}
F(\mathcal{Q}, T) &= \Omega(\mu, T) + \mu \mathcal{Q} \\
&= -T \left[\frac{1}{2} \ln \left(\frac{4}{1 - 4\mathcal{Q}^2} \right) + \mathcal{Q} \ln \left(\frac{1 - 2\mathcal{Q}}{1 + 2\mathcal{Q}} \right) \right] \\
&\quad - 2\pi T(1 - 4\mathcal{Q}^2) \left[\tan \left(\frac{\pi v}{2} \right) - \frac{\pi v}{4} \right] \Delta^2 + \mathcal{O}(\Delta^3).
\end{aligned} \tag{C19}$$

It is more convenient to work with the canonical $F(\mathcal{Q}, T)$, rather than the grand canonical $\Omega(\mu, T)$, because \mathcal{Q} is universal, while μ is not. We will use (C19) to verify the universal expressions in Section II C 1, and also to obtain new non-universal results.

First, in the fixed \mathcal{Q} ensemble, we can compute the chemical potential $\mu(\mathcal{Q}, T)$ needed to keep \mathcal{Q} fixed. We find

$$\begin{aligned}
\mu(\mathcal{Q}, T) &= \left(\frac{\partial F}{\partial \mathcal{Q}} \right)_T \\
&= -T \ln \left(\frac{1 - 2\mathcal{Q}}{1 + 2\mathcal{Q}} \right) + 16\pi T \mathcal{Q} v \left[\tan \left(\frac{\pi v}{2} \right) - \frac{\pi v}{4} \right] \Delta^2 + \mathcal{O}(\Delta^3) \\
&= \mu_0 - 2\pi \mathcal{E}(\mathcal{Q}) T + \mathcal{O}(T^2)
\end{aligned} \tag{C20}$$

In the last line, we have taken the low T limit at fixed \mathcal{J} using (C13), and we find precisely the expression (2.14), with the universal function $\mathcal{E}(\mathcal{Q})$ given by (C2), and the non-universal bare chemical potential

$$\mu_0 = 16\mathcal{J}\mathcal{Q}\Delta^2 + \mathcal{O}(\Delta^3). \tag{C21}$$

Note that there is no $\mathcal{O}(\Delta^0)$ term in μ_0 : this has consequences for the compressibility. From (C20) we can obtain the inverse compressibility, $1/K$, by taking a derivative w.r.t. \mathcal{Q} ; at low T we have

$$\frac{1}{K} = \left(\frac{\partial \mu}{\partial \mathcal{Q}} \right)_T = \frac{4T}{1 - 4\mathcal{Q}^2} + (16\mathcal{J} - 4\pi^2 T) \Delta^2 + \mathcal{O}(\Delta^3) \tag{C22}$$

So we now see that if take the limit $\Delta \rightarrow 0$ first, then the compressibility diverges as $K \sim 1/T$ in the low T limit at fixed \mathcal{J} . On the other hand, if we take the $T \rightarrow 0$ at non-zero Δ , then K remains finite at $K = q^2/(16\mathcal{J})$, as needed for the consistency of the analysis in Section II D. Note that the large q expansion holds for $1/K$, and not for K , it is the expansion for $1/K$ which establishes the finiteness of K as $T \rightarrow 0$.

We can also obtain the non-universal ground state energy

$$E_0 = F(\mathcal{Q}, T \rightarrow 0) = -2\mathcal{J}(1 - 4\mathcal{Q}^2)\Delta^2 + \mathcal{O}(\Delta^3). \tag{C23}$$

This is compatible with (C21) and (2.15).

Finally, we can compute the entropy, and perform its low T expansion at fixed \mathcal{J} ; we find

$$\begin{aligned}\mathcal{S}(\mathcal{Q}, T) &= - \left(\frac{\partial F}{\partial T} \right)_{\mathcal{Q}} \\ &= \mathcal{S}(\mathcal{Q}) + \gamma T + \mathcal{O}(T^2),\end{aligned}\tag{C24}$$

where the universal function $\mathcal{S}(\mathcal{Q})$ agrees with (C1), and the non-universal linear-in- T coefficient of the specific heat at fixed \mathcal{Q} is

$$\gamma = \frac{2\pi^2(1 - 4\mathcal{Q}^2)}{\mathcal{J}} \Delta^2 + \mathcal{O}(\Delta^3)\tag{C25}$$

Again, note that there is no $\mathcal{O}(\Delta^0)$ term in γ .

Appendix D: Luttinger-Ward analysis

The appendix will generalize the Luttinger-Ward analysis in Appendix A of Ref. 35 (hereafter referred to as GPS) from $q = 4$ to general q . The Luttinger-Ward (LW) functional for general q reads:

$$\Phi[G] = -J^2 (-1)^{q/2} \int d\tau G(\tau)^{q/2} G(-\tau)^{q/2}\tag{D1}$$

such that:

$$\Sigma(\tau) = \frac{\delta \Phi}{\delta G(-\tau)} = -(-1)^{q/2} q J^2 G(\tau)^{q/2} G(-\tau)^{q/2-1}\tag{D2}$$

in accordance with Eq. (2.1).

The low frequency Green's function ansatz in Eq. (2.3) has a prefactor C given in Eq. (2.5). Here, we write the prefactor as

$$C(q, \theta)^q = K(q) [s_+ s_-]^{(2-q)/2}\tag{D3}$$

in which $K(q)$ only depends on q , and we will use the notation $s_{\pm} \equiv \sin(\pi/q \pm \theta)$. We also note that Eq. (2.3) implies the following low-frequency behaviour of the spectral function:

$$A(\omega) \equiv -\frac{1}{\pi} \text{Im} G(\omega + i0^+) = \frac{C}{\pi} \frac{s_{\pm}}{|\omega|^{1-2/q}}\tag{D4}$$

in which the $+$ ($-$) sign applies to positive (negative) frequencies respectively. Hence, the spectral asymmetry is given by: $A(\omega)/A(-\omega) = s_+/s_-$.

We proceed along the lines of Appendix A of GPS. Eq. (A4) is unchanged and reads:

$$\mathcal{Q} = -\frac{\theta}{\pi} - i \int \frac{d\omega}{2\pi} G^F(\omega) \partial_{\omega} \Sigma^F(\omega) e^{i\omega 0^+},\tag{D5}$$

where the superscript F indicates Feynman Green's functions at $T = 0$ and real frequency. We will actually not perform a fully explicit calculation of the integral on the r.h.s ('anomalous' term) but instead obtain its value from a simple argument. This argument is the one on page 14 (bottom of first column) of GPS, and it turns out that it can be generalized to arbitrary q .

Imagine one makes an explicit calculation of the anomalous term, along the lines of Appendix A of GPS. Then, one would have a sum of terms which all involve a product of q spectral functions because the LW functional is a polynomial of degree q in G . The spectral functions can either be for negative or positive frequency (see Eq. (A.11) in GPS) and hence at the end of the computation, using the low-frequency form (D4) we get a sum of terms:

$$C^q \sum_{n=0}^{q/2} c_n \left[s_+^{q/2+n} s_-^{q/2-n} - s_+^{q/2-n} s_-^{q/2+n} \right] \quad (\text{D6})$$

We have used the fact that this must be an odd function of θ (hence the antisymmetry) and have assumed that the anomalous term only depends on the IR properties (this is the part which needs a detailed proof by regularisation as in GPS). The coefficients c_n depend a priori on q but not on θ because all θ -dependence is contained in $C(q, \theta)$ and s_{\pm} .

Let us examine these terms. The $n = 0$ one vanishes by symmetry. The $n = 1$ term yields:

$$c_1 C^q \left[s_+^{q/2+1} s_-^{q/2-1} - s_+^{q/2-1} s_-^{q/2+1} \right] = c_1 C^q (s_+ s_-)^{q/2-1} (s_+^2 - s_-^2) \quad (\text{D7})$$

Using Eq. (D3), this simplifies to:

$$c_1 K(q) (s_+^2 - s_-^2) = c_1 K(q) \sin\left(\frac{2\pi}{q}\right) \sin 2\theta \quad (\text{D8})$$

The important point here is that the only θ -dependence is in the $\sin 2\theta$ term.

Let us now consider the terms with $n > 1$. It is easily seen that *all* these terms involve a combination of s_{\pm} which has a *divergence* at either $\theta = \pi/q$ or $-\pi/q$, the reason being that the factor $[s_+ s_-]^{(2-q)/2}$ in the prefactor C^q no longer cancels (note that $2 - q < 0$). These terms are not admissible because at $\theta = \pm\pi/q$, the fermion occupation number should either vanish or go to unity, and cannot diverge. Hence, these terms should not appear and all c_n 's with $n > 1$ should be zero. For $q = 4$, the only such term is $C^4(s_+^4 - s_-^4)$, which we eliminated for the same reason in the heuristic argument of Appendix A of GPS. It extends here to all $n = 2, \dots, q/2$.

As a result, this argument shows that a full calculation of the anomaly will yield (with a simple redefinition of c_1):

$$\mathcal{Q} = -\frac{\theta}{\pi} - \tilde{c}_1(q) \sin 2\theta \quad (\text{D9})$$

Fixing the constant is straightforward: we note that for $\theta = +\pi/q$ the negative-frequency spectral function vanishes and thus we should get the smallest fermion number ($\mathcal{Q} = -1/2$). Hence

$$-\frac{1}{2} = -\frac{1}{q} - \tilde{c}_1 \sin \frac{2\pi}{q}, \quad (\text{D10})$$

and we finally obtain Eq. (2.24).

Appendix E: Numerical solution of the SYK model

This appendix describes our numerical solution of Eqs. (2.1) and (2.2) for the case $q = 4$.

We worked in the frequency domain by writing Eq. (2.1) as a convolution

$$\Sigma(i\omega_n) = -\frac{J^2}{\beta^2} \sum_{\omega_n = \omega_1 + \omega_2 - \omega_3} G(i\omega_1)G(i\omega_2)G(i\omega_3) \quad (\text{E1})$$

We used the function package `conv_fft2` in Matlab to perform the convolution. We restricted the frequency argument in $G(i\omega_n)$ to be $2\pi T(n + \frac{1}{2})$ where $-N \leq n \leq N - 1$. After the convolution, we cut off the frequency argument in $\Sigma(i\omega_n)$ to be within the same regime. Finally, we updated the Green's function in a weighted way:

$$G_j(i\omega_n) = (1 - \alpha)G_j(i\omega_n) + \alpha \frac{1}{i\omega_n + \mu - \Sigma_{j-1}(i\omega_n)} \quad (\text{E2})$$

where we choose the weight $\alpha = 0.2$, and j denotes the iteration step.

We also used a second numerical approach in which we directly evaluate Eqs. (2.1) and (2.2) in frequency space and time space separately, and then use fast Fourier transforms (FFT) between them. But there is a subtlety: when considering the transformation from τ space to ω_n space, we are doing a discrete sum to represent the numerical integral. For a sensible discrete sum, we do not want the exponential phase to vary too much between the two adjacent discrete points. So we want $\omega_n(\tau_j - \tau_{j-1}) \ll 1$. With N_τ and N_ω the number of points of τ and ω , we need $N_\omega/N_\tau \ll 1$. We found $N_\omega = 2^{18}$, $N_\tau = 2^{20}$ gave accurate results.

From the numerical solution for Green's function and self energy, we obtain the grand potential by evaluating Eq. (C14). In practice, we want to subtract the grand potential of a free theory and then add it back to obtain a convergent sum over frequencies. So we wrote the first term in Eq. (C14) as

$$T \sum_n \log [G(i\omega_n)/G_0(i\omega_n)] + T \log [1 + e^{\mu/T}]. \quad (\text{E3})$$

By the equations of motion, the second integral can be written as

$$-\frac{3}{4\beta} \sum_n \Sigma(i\omega_n)G(i\omega_n) \quad (\text{E4})$$

Then we put the solution into these two terms and obtained the grand potential $\Omega(\mu, T)$. The density, \mathcal{Q} , the compressibility, K , and the entropy, \mathcal{S} were then obtained from suitable thermodynamic derivatives⁶. Our numerical results for $\mathcal{S}(\mathcal{Q})$, obtained by both methods are shown in

⁶ \mathcal{Q} can also be obtained from $G(\tau = 0^-)$, we have checked that it is consistent with the derivative method

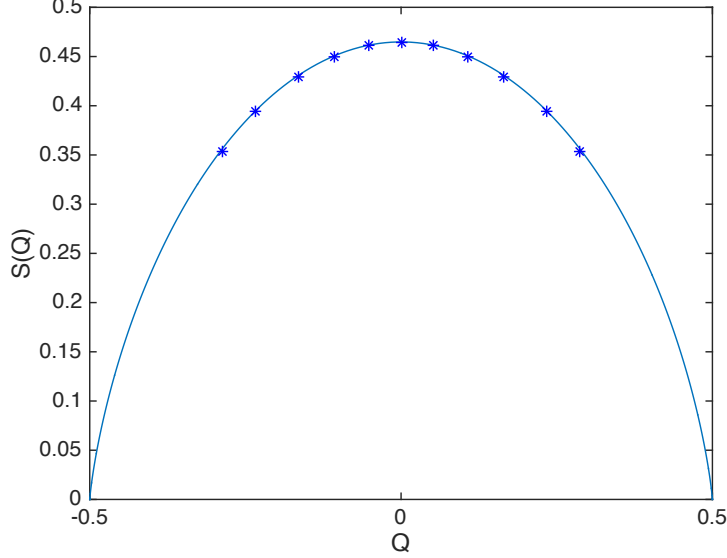


FIG. 5. The entropy $\mathcal{S}(\mathcal{Q})$ obtained from the exact results [35] in Section II C 3 (full line), and by the numerical solutions (stars).

Fig. 5, they are in excellent agreement with the exact analytic results [35]. In the frequency domain computation, we used the cutoff $N = 2 \times 10^6$. The points in Fig. 5 are at moderate values of \mathcal{Q} , and our numerics did not converge for $|\mathcal{Q}|$ near $1/2$.⁷

For the compressibility, numerically near $\mu = 0$ and at $T = 0$, we find that $K = 1.04/J = 1.04/(\sqrt{2}\mathcal{J})$; With $q = 4$, this is of the same order of the large q result: $K = q^2/(16\mathcal{J}) = 1/\mathcal{J}$.

Appendix F: Normal mode analysis of the SYK model

This appendix will generalize the analysis of Maldacena and Stanford [47], and describe the structure of the effective action for fluctuations directly from the action in Eq. (C14). We will work here in an angular variable

$$\varphi = 2\pi T\tau \quad (\text{F1})$$

which takes values on a temporal circle of unit radius. We also use the notation $\varphi_{12} \equiv \varphi_1 - \varphi_2$.

We begin with the saddle-point solution of Eq. (C14), the Green's function $G_s(\varphi)$. In the scaling

⁷ At large μ , we always find the free Green's function $G_0 = \frac{1}{i\omega_n + \mu}$ to be solution. The reason can be understood by the self-energy obtained from the free solution

$$\Sigma_0(i\omega_n) = -\frac{J^2}{\beta^2} \sum_{\omega_n = \omega_1 + \omega_2 - \omega_3} G_0(i\omega_1)G_0(i\omega_2)G_0(i\omega_3) = -\frac{J^2}{i\omega_n + \mu} \frac{1}{(2 \cosh \frac{\beta\mu}{2})^2}$$

Notice the exponential suppression at low temperature. This means at any finite μ , at zero temperature, the free one is always a solution. Numerically we are always at small finite temperature to represent the zero temperature result, but when μ becomes large, the exponential suppression will make the free Green's function converge well within the fixed tolerance.

limit, this is given by Eq. (2.9). We write this here as

$$G_s(\varphi) = b \frac{e^{-\mathcal{E}\varphi}}{(\sin \frac{\varphi}{2})^{2\Delta}}, \quad \Delta = \frac{1}{q}, \quad \varphi \in [0, 2\pi) \quad (\text{F2})$$

where the prefactor b is specified in Eq. (2.9). We now expand the effective action Eq. (C14) to quadratic order of the fluctuations $\delta G(\varphi_1, \varphi_2) = G(\varphi_1, \varphi_2) - G_s(\varphi_{12})$, $\delta \Sigma(\varphi_1, \varphi_2) = \Sigma(\varphi_1, \varphi_2) - \Sigma_s(\varphi_{12})$ and further integrate over $\delta \Sigma$. For convenience we use renormalized form of the fluctuation:

$$g(\varphi_1, \varphi_2) = [-G_s(\varphi_{12})G_s(\varphi_{21})]^{q/4} G_s(\varphi_{12})^{-1} \delta G(\varphi_1, \varphi_2) \quad (\text{F3})$$

and obtaining the action (to quadratic order) in g :

$$\frac{S_{\text{eff.}}}{N} = \frac{1}{2} \int d^4\varphi g(\varphi_1, \varphi_2) Q(\varphi_1, \varphi_2; \varphi_3, \varphi_4) g(\varphi_3, \varphi_4), \quad (\text{F4})$$

where Q is a quadratic form on the space of functions with two time variables.

We now focus on just the zero mode fluctuations specified by the transformations in Eq. (2.27). First, examine the infinitesimal reparameterization mode, with an accompanying U(1) transformation satisfying Eq. (2.31)

$$f(\varphi) = \varphi + \epsilon(\varphi) \quad , \quad \phi(\varphi) = -i\mathcal{E}\epsilon(\varphi). \quad (\text{F5})$$

Notice that under this mode, $\tilde{\phi} = 0$ in Eq. (2.36). Inserting Eq. (F5) into Eq. (2.27), using Eq. (F3) to get the normalized fluctuations for each Fourier mode $\epsilon(\varphi) = \frac{1}{2\pi} \sum_n e^{-in\varphi} \epsilon_n$, we find that the linear order change in g is

$$g_n^\epsilon(\varphi_1, \varphi_2) = \frac{i\Delta b^{q/2} e^{-2\pi q\mathcal{E}/4}}{\pi} \left(\frac{f_n(\varphi_{12})}{|\sin \frac{\varphi_{12}}{2}|} \right) \epsilon_n e^{-in\frac{\varphi_1+\varphi_2}{2}}. \quad (\text{F6})$$

The functions $f_n(\varphi_{12})$ is a symmetric function of two variables φ_1, φ_2 :

$$f_n(\varphi) = \frac{\sin n\frac{\varphi}{2}}{\tan \frac{\varphi}{2}} - n \cos n\frac{\varphi}{2} \quad , \quad \int_0^{2\pi} d\varphi \left(\frac{f_n(\varphi)}{\sin \frac{\varphi}{2}} \right)^2 = \frac{2\pi}{3} |n|(n^2 - 1). \quad (\text{F7})$$

Similarly, we can examine the U(1) fluctuation mode, under which ϵ is unchanged but ϕ changes:

$$g_n^\phi(\varphi_1, \varphi_2) = \frac{b^{q/2} e^{-2\pi q\mathcal{E}/4}}{\pi} \left(\frac{\sin n\frac{\varphi_{12}}{2}}{|\sin \frac{\varphi_{12}}{2}|} \right) \phi_n e^{-in\frac{\varphi_1+\varphi_2}{2}}, \quad (\text{F8})$$

which implies that the phase fluctuation is anti-symmetric in two time variable. It is also useful to notice the following equation:

$$\int_0^{2\pi} d\varphi \left(\frac{\sin n\frac{\varphi}{2}}{\sin \frac{\varphi}{2}} \right)^2 = 2\pi |n|. \quad (\text{F9})$$

Turning to the structure of the quadratic form, Q , we now make the key observation that evaluating Q from Eq. (C14) and the conformal Green's function in Eq. (F2) leads to a vanishing action of Q on the normal modes described above. This is a direct consequence of the invariance of Eq. (2.27) under reparameterization and $U(1)$ transformations. Ref. 47 argued that going beyond the conformal limit will lead to a shift in the eigenvalue of Q of order $|n|T/J$ in the first order perturbation theory. Assuming this applies here to both modes discussed above,⁸ we have

$$Q \cdot g_n^\phi = \alpha_\phi \frac{|n|T}{J} g_n^\phi, \quad Q \cdot g_n^\epsilon = \alpha_\epsilon \frac{|n|T}{J} g_n^\epsilon, \quad (\text{F10})$$

where the numerical coefficients α_ϕ and α_ϵ cannot be obtained analytically, but can be computed in the large q expansion. Here, we can fix them by comparing with the large q results already obtained in Appendix C.

Inserting Eq. (F10) into (F4), and using the explicit form of the fluctuations in Eqs. (F6) and (F8), we obtain the effective action to quadratic order:

$$\frac{S_{\text{eff.}}}{N} = \frac{1}{2} \sum_n \left\{ c_\phi n^2 |\tilde{\phi}_n|^2 + c_\epsilon n^2 (n^2 - 1) |\epsilon_n|^2 \right\}. \quad (\text{F11})$$

where c_ϕ and c_ϵ are coefficients of order $\frac{T}{J}$ and proportional to α_ϕ and α_ϵ . We confirm that this is of the form in Eq. (2.37), and we can further express the ratio of K and γ in terms of the numerical coefficients here

$$\frac{K}{\gamma} = \frac{c_\phi}{4\pi^2 c_\epsilon} = \frac{3\alpha_\phi}{4\pi^2 \Delta^2 \alpha_\epsilon} \quad (\text{F12})$$

Using the effective action Eq. (F11) we can also extract an order-one piece of the free energy which arises from the 1-loop calculation. In addition to the Schwarzian part that has been discussed in Ref. [47], we have a new piece from phase fluctuations $\tilde{\phi}$:

$$Z_{\tilde{\phi}}(\beta) = \sqrt{\det B}^{-1}, \quad B_{n,m} = \delta_{n+m} \frac{N c_\phi}{2} n^2 \quad (\text{F13})$$

We can evaluate the determinant using the zeta function regularization :

$$\log Z_{\tilde{\phi}} = - \left(\sum_{n=1}^{\infty} \log \frac{N c_\phi}{2} n^2 \right) = \frac{1}{2} \log \frac{N c_\phi}{8\pi^2} \sim -\frac{1}{2} \log \beta J \quad (\text{F14})$$

Together with the contribution from Schwarzian ($\sim -\frac{3}{2} \log \beta J$), we conclude that the partition function $Z(\beta)$ is proportional to β^{-2} at large β . From this, one can further extract the low energy density of state $\rho(E)$ from inverse Laplace transformation of $Z(\beta)$, and show that $\rho(E)$ is proportional to E at small $E \ll \frac{J}{N}$.

⁸ One can justify this statement by a renormalization theory argument[89]

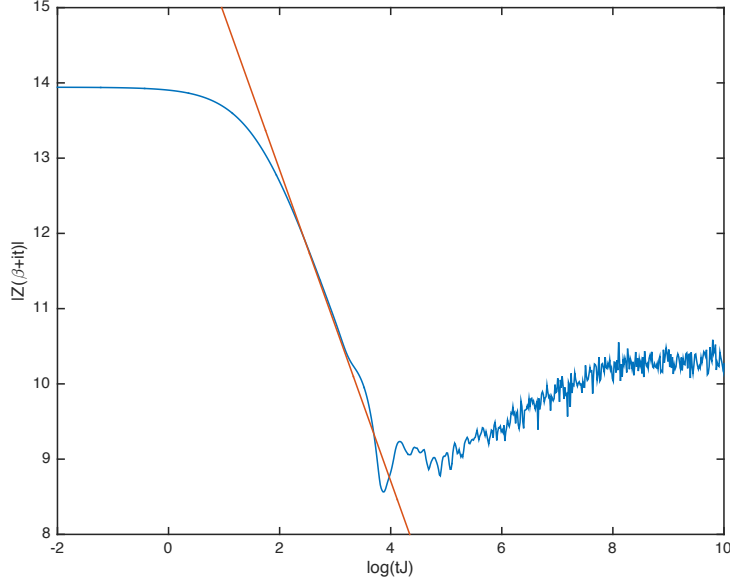


FIG. 6. Blue line is $|Z(\beta + it)|$ computed from exact diagonalization data for $N = 15$ and 1000 realizations at $\beta J = 5$. The orange line is the linear fit for the "slope" regime and the slope is around -2.07 .

We have also numerically computed a variation of partition function $|Z(\beta + it)|$ as in Ref. [62] using exact diagonalization, the result is shown in Fig. 6. The slope is around -2.07 in the "slope" regime which is naively outside the validity of the one-loop computation ($1 \ll |\beta + it| \ll N$), this is an indication of the one-loop exactness[90] of the complex SYK model.

Appendix G: Couplings in effective action of the SYK model

This appendix will present another derivation for the values of the couplings in the Schwarzian and phase fluctuation effective action in Eq. (2.33). Here, we will only obtain the leading quadratic terms in the gradient expansion, which have two temporal derivatives, although Eq. (2.33) contains many higher order terms. Just by matching these low order terms we will fix the couplings as in Eq. (2.42).

First we examine phase fluctuations, under which by Eq. (2.27)

$$\begin{aligned} G(\tau, \tau') &= e^{i\phi(\tau)} G_s(\tau - \tau') e^{-i\phi(\tau')} \\ \Sigma(\tau, \tau') &= e^{i\phi(\tau)} \Sigma_s(\tau - \tau') e^{-i\phi(\tau')} \end{aligned} \tag{G1}$$

We insert the ansatz (G1) into the action (C14), and perform a gradient expansion in derivatives of $\phi(\tau)$. It is evident that the entire contribution comes from the $\text{Tr} \ln$ term, as the other terms

are independent of ϕ . Furthermore, we can use the identity

$$\begin{aligned} \text{Tr} \ln \left[\delta(\tau - \tau') \left(-\frac{\partial}{\partial \tau} + \mu \right) - e^{i\phi(\tau)} \Sigma_s(\tau - \tau') e^{-i\phi(\tau')} \right] \\ = \text{Tr} \ln \left[\delta(\tau - \tau') \left(-\frac{\partial}{\partial \tau} + \mu + i\partial_\tau \phi(\tau) \right) - \Sigma_s(\tau - \tau') \right], \end{aligned} \quad (\text{G2})$$

which is easily derived by a gauge transformation of the fermion fields that were integrated to obtain the determinant. In a gradient expansion about a saddle point at a fixed μ , after all other modes (other than the reparameterization mode mentioned below) have been integrated out, we expect an effective action of the form

$$\frac{S_\phi}{N} = \frac{K}{2} \int_0^{1/T} d\tau (\partial_\tau \phi)^2. \quad (\text{G3})$$

We can determine K by evaluating the effective action for the special case where $\partial_\tau \phi$ a constant; under these conditions, we note from (G2) that all we have to do in the effective action is to make a small change in μ by $i\partial_\tau \phi$. Therefore, we have established that

$$K = - \left(\frac{\partial^2 \Omega}{\partial \mu^2} \right)_T \quad (\text{G4})$$

is indeed the compressibility, as in Eq. (2.42).

A similar argument can made for energy fluctuations. Now we consider the temporal reparameterization

$$\tau \rightarrow \tau + \epsilon(\tau) \quad (\text{G5})$$

After integrating out all other high energy modes at a fixed chemical potential (other than the phase mode above), we postulate an effective action for $\epsilon(\tau)$, and assume that the lowest order gradient expansion leads to

$$\frac{S_\epsilon}{N} = \frac{\tilde{K}}{2} \int_0^{1/T} d\tau (\partial_\tau \epsilon)^2. \quad (\text{G6})$$

We can now relate the coefficient to a thermodynamic derivative. As for (G3), consider the case where $\partial_\tau \epsilon$ is a constant. Then (G5) implies a change in temperature

$$-\frac{\delta T}{T^2} = \frac{\partial_\tau \epsilon}{T} \quad (\text{G7})$$

Inserting (G7) into (G6), we conclude that

$$\frac{\tilde{K}}{T^2} = \left(\frac{\partial^2 \Omega}{\partial T^2} \right)_\mu \quad (\text{G8})$$

Finally, we can also fix the cross term by a similar argument, and so obtain the complete Gaussian effective action for ϕ and ϵ fluctuations, after all other modes have been integrated out

$$\frac{S_{\phi\epsilon}}{N} = \int_0^{1/T} d\tau \left[-\frac{1}{2} \left(\frac{\partial^2 \Omega}{\partial \mu^2} \right)_T (\partial_\tau \phi)^2 - iT \frac{\partial^2 \Omega}{\partial T \partial \mu} (\partial_\tau \epsilon)(\partial_\tau \phi) + \frac{T^2}{2} \left(\frac{\partial^2 \Omega}{\partial T^2} \right)_\mu (\partial_\tau \epsilon)^2 \right]. \quad (\text{G9})$$

After application of thermodynamic identities, this is found to agree with the second order temporal derivatives in Eq. (2.33), and the identifications in Eqs. (1.7) and (2.42).

Appendix H: Diffusion constants of the higher-dimensional SYK model

The generalization of the zero-dimensional SYK results in Appendix F to the higher dimensional models closely follows the lines of Ref. 57. In high dimensional models, the quadratic form Q acquires a spatial dependence, formally we have $Q \rightarrow Q_{xy}$ where Q_{xy} contains a hopping matrix for the fluctuations, which can be easily diagonalized by going to k -space. For long wavelength limit, we can expand its eigenvalue around $k = 0$: $Q(k) = Q(0) + ck^2 + \dots$ where c is a constant depends on J_0 and J_1 that captures the band structure at long wavelength, and $Q(0)$ is the quadratic form at $k = 0$ which reproduces the quadratic form in $(0+1)$ -dimension. In general, the hopping matrix acts differently on anti-symmetric fluctuation g^ϕ and symmetric fluctuation g^ϵ , which will induce two different band structures $Q(k)^\phi = Q(0)^\phi + c_1 k^2 + \dots$ and $Q(k)^\epsilon = Q(0)^\epsilon + c_2 k^2 + \dots$ for charge and energy fluctuation respectively.⁹

Inserting this back into the effective action derivation in Appendix F, we notice that for the ϕ modes, we need to replace the UV correction for Q from $Q^\phi(0) \sim \alpha_\phi \frac{|n|T}{J}$ to $Q^\phi(k) = \alpha_\phi \frac{|n|T}{J} + c_1 k^2$. Similarly, for ϵ modes, we need to replace $\alpha_\epsilon \frac{|n|T}{J}$ to $\alpha_\epsilon \frac{|n|T}{J} + c_2 k^2$, where $J = \sqrt{J_0^2 + J_1^2}$. This replacement leads to the effective action in Eq. (2.47) with

$$D_1 = \frac{2\pi c_1 J}{\alpha_\phi} \quad , \quad D_2 = \frac{2\pi c_2 J}{\alpha_\epsilon}. \quad (\text{H1})$$

For the specific model we discussed in main text, the special form of the hopping term Eq. (2.45) leads to $c_1 = c_2$. Using Eq. (F12), we then obtain the ratio of the diffusion constants

$$\frac{D_2}{D_1} = \frac{\alpha_\phi}{\alpha_\epsilon} = \frac{4\pi^2 \Delta^2}{3} \frac{K}{\gamma} \quad (\text{H2})$$

which was presented in Eq. (2.48).

⁹ This is different from the SYK with Majorana fermions, where we have symmetries in Green's function when exchanging two time variables. More details about the properties of the fluctuations in complex SYK model will be discussed in Ref. [91].

Appendix I: More general AdS₂ solutions

The field theory dual to the solution (3.2) shares the property (3.34) with the SYK model because of the AdS₂ factor in its near-horizon geometry. To further validate this, this Appendix will look at more complicated gravitational theories which also have solutions that break translational symmetry and have AdS₂ factors in their near-horizon geometry. The UV details of these differ from those of the solution (3.2), but we will find that the relation (1.13) is nevertheless obeyed. We will consider only homogeneous solutions for which we can write down analytic solutions. It would be interesting to see how far this result generalizes, particularly to cases where translational symmetry is broken inhomogeneously.

1. Asymptotically AdS₄

We will study a more general class of gravitational actions than (3.1), by including a new scalar field Φ in the four dimensional action. By choosing the potential and the gauge field coupling appropriately, one can find a whole class of solutions which are asymptotically AdS₄ and have a near-horizon AdS₂ geometry [26]. The action is

$$S = \int d^4x \sqrt{-g} \left(\mathcal{R} - \frac{1}{2}(\partial\Phi)^2 + V(\Phi) - \frac{1}{2} \sum_{i=1}^2 (\partial\varphi_i)^2 - \frac{Z(\Phi)}{4} F_{\mu\nu} F^{\mu\nu} \right), \quad (\text{I1})$$

where $V(\Phi)$ and $Z(\Phi)$ are a family of functions depending on a single parameter δ

$$V(\Phi) = \frac{16\delta^2}{(1+\delta^2)^2} e^{\frac{(\delta^2-1)}{2\delta}\Phi} + \frac{2(3-\delta^2)}{(1+\delta^2)^2} e^{\delta\Phi} - \frac{2(1-3\delta^2)}{(1+\delta^2)^2} e^{-\Phi/\delta}, \quad Z(\Phi) = e^{-\delta\Phi}, \quad (\text{I2})$$

The asymptotically AdS₄ solutions which break translational symmetry homogeneously are¹⁰

$$\begin{aligned} ds^2 &= -r^2 f(r) h(r)^{-\frac{2}{1+\delta^2}} dt^2 + \frac{dr^2}{r^2 f(r) h(r)^{-\frac{2}{1+\delta^2}}} + r^2 h^{\frac{2}{1+\delta^2}} d\vec{x}^2, \quad \varphi_i = m x^i, \\ f(r) &= h(r)^{\frac{4}{1+\delta^2}} - \frac{r_0^3}{r^3} h(r_0)^{\frac{4}{1+\delta^2}} - \frac{m^2}{2r^2} \left(1 - \frac{r_0}{r}\right), \quad h(r) = 1 + \frac{Q}{r}, \\ A_t(r) &= 2\sqrt{\frac{Qr_0}{1+\delta^2} \left(h(r_0)^{2\frac{1-\delta^2}{1+\delta^2}} - \frac{m^2}{2r_0^2 h(r_0)} \right) \frac{(1 - \frac{r_0}{r})}{h(r)}}, \quad \Phi = -\frac{2\delta}{1+\delta^2} \log h(r). \end{aligned} \quad (\text{I3})$$

When $\delta = 0$, the scalar field vanishes, and the solution reduces to (3.2). When $\delta = 1/\sqrt{3}$, the solution reduces to that studied in [13], which is conformal to AdS₂ \times \mathbb{R}^2 near the horizon at zero temperature, and has a linear-in-temperature entropy at small T . For any value $0 \leq \delta < 1/\sqrt{3}$,

¹⁰ There are typos in the solutions and thermodynamic properties written in [26], which have been corrected here.

the solution has a near-horizon $\text{AdS}_2 \times \mathbb{R}^2$ geometry at zero temperature, and we will restrict to this parameter range from now on, assuming that both Q and r_0 are positive. The thermodynamic properties of this solution are

$$\begin{aligned} T &= \frac{r_0}{4\pi} h(r_0)^{-\frac{2}{1+\delta^2}} \left(\frac{4 - (1 - 3\delta^2)h(r_0)}{1 + \delta^2} h(r_0)^{\frac{3-\delta^2}{1+\delta^2}} - \frac{m^2}{2r_0^2} \right), & \mathcal{S} &= 4\pi r_0^2 h(r_0)^{\frac{2}{1+\delta^2}}, \\ \mu &= 2\sqrt{\frac{Qr_0}{1 + \delta^2} \left(h(r_0)^{2\frac{1-\delta^2}{1+\delta^2}} - \frac{m^2}{2r_0^2 h(r_0)} \right)}, & \mathcal{Q} &= \mu(Q + r_0), \end{aligned} \quad (\text{I4})$$

so that $T = 0$ corresponds to the condition

$$\frac{4 - (1 - 3\delta^2)h(r_0)}{1 + \delta^2} h(r_0)^{\frac{3-\delta^2}{1+\delta^2}} = \frac{m^2}{2r_0^2}. \quad (\text{I5})$$

To obtain the $T = 0$ near-horizon geometry, one should perform the coordinate transformation

$$\zeta = \frac{r - r_*}{\epsilon}, \quad \tau = \epsilon t, \quad (\text{I6})$$

followed by the small ϵ limit to give

$$\begin{aligned} ds^2 &= \left(-\frac{\zeta^2}{\tilde{L}^2} d\tau^2 + \frac{\tilde{L}^2}{\zeta^2} d\zeta^2 \right) + r_0^2 h(r_0)^{\frac{2}{1+\delta^2}} d\vec{x}^2 + O(\epsilon), \\ A_\tau &= \frac{\mathcal{E}}{\tilde{L}^2} \zeta + O(\epsilon), \quad \varphi_i = m x^i, \quad \Phi = -\frac{2\delta}{1 + \delta^2} \log(h(r_0)). \end{aligned} \quad (\text{I7})$$

The AdS_2 radius of curvature is

$$\tilde{L}^2 = \frac{(1 + \delta^2)^2 h(r_0)^{\frac{2\delta^2}{1+\delta^2}}}{2(3 - \delta^2) - h(r_0)(1 - 3\delta^2)(4 - h(r_0)(1 - \delta^2))}, \quad (\text{I8})$$

and the appropriately normalized AdS_2 electric field in these units is

$$\mathcal{E} = \tilde{L}^2 A'_\tau(\zeta) = 2(1 + \delta^2) \frac{\sqrt{(h(r_0) - 1)(-(3 - \delta^2) + h(r_0)(1 - 3\delta^2))}}{2(3 - \delta^2) - h(r_0)(1 - 3\delta^2)(4 - h(r_0)(1 - \delta^2))}. \quad (\text{I9})$$

The dc conductivities are given by properties of the solution at the horizon [23, 80]. Explicitly, they are

$$\sigma = \frac{4\pi \mathcal{Q}^2}{m^2 \mathcal{S}} + Z(\Phi(r_0)), \quad \alpha = \frac{4\pi \mathcal{Q}}{m^2}, \quad \bar{\kappa} = \frac{4\pi \mathcal{S} T}{m^2}. \quad (\text{I10})$$

In the limit $T \rightarrow 0$, the Seebeck coefficient is

$$S \equiv \frac{\alpha}{\sigma} = 4\pi (1 + \delta^2) \frac{\sqrt{(h(r_0) - 1)(-(3 - \delta^2) + h(r_0)(1 - 3\delta^2))}}{2(3 - \delta^2) - h(r_0)(1 - 3\delta^2)(4 - h(r_0)(1 - \delta^2))} = 2\pi \mathcal{E}. \quad (\text{I11})$$

To verify the relation (1.13), we require the thermodynamic susceptibilities of these solutions. In general, it's not possible to invert (I4) to obtain closed form expressions for $\mathcal{S}(\mathcal{Q}, T, m)$ etc. It is convenient to use $r_0, h(r_0)$ and m as our independent parameters, such that (for fixed m),

$$\delta\mathcal{S} = \frac{\partial\mathcal{S}}{\partial r_0} \Big|_{h(r_0)} \delta r_0 + \frac{\partial\mathcal{S}}{\partial h(r_0)} \Big|_{r_0} \delta h(r_0), \quad (\text{I12})$$

and similarly for other thermodynamic objects. Variations at fixed T therefore correspond to the condition

$$\delta h(r_0) = - \frac{\partial T}{\partial r_0} \Big|_{h(r_0)} \left(\frac{\partial T}{\partial h(r_0)} \Big|_{r_0} \right)^{-1} \delta r_0, \quad (\text{I13})$$

and so the relevant thermodynamic susceptibility can be written

$$\left(\frac{\partial\mathcal{S}}{\partial\mathcal{Q}} \right)_T = \frac{\frac{\partial T}{\partial h(r_0)} \Big|_{r_0} \frac{\partial\mathcal{S}}{\partial r_0} \Big|_{h(r_0)} - \frac{\partial\mathcal{S}}{\partial h(r_0)} \Big|_{r_0} \frac{\partial T}{\partial r_0} \Big|_{h(r_0)}}{\frac{\partial T}{\partial h(r_0)} \Big|_{r_0} \frac{\partial\mathcal{Q}}{\partial r_0} \Big|_{h(r_0)} - \frac{\partial\mathcal{Q}}{\partial h(r_0)} \Big|_{r_0} \frac{\partial T}{\partial r_0} \Big|_{h(r_0)}}. \quad (\text{I14})$$

Evaluating this in the limit $T \rightarrow 0$ gives

$$S = \left(\frac{\partial\mathcal{S}}{\partial\mathcal{Q}} \right)_T = 2\pi\mathcal{E}, \quad (\text{I15})$$

in agreement with (1.13). For any non-zero m , the low energy correlators of the dual field theory should be those of diffusive hydrodynamics, and because of the relation (1.13), they will have the same form (2.49) as those of the higher dimensional SYK model in the small T limit, with the parameters

$$\begin{aligned} r_0 D_1 &= h(r_0)^{-\frac{2}{1+\delta^2}} \frac{2(3-\delta^2) - h(r_0)(1-3\delta^2)(3-\delta^2(1-2h(r_0)))}{(2(1-\delta^2) - h(r_0)(1-3\delta^2))(4-h(r_0)(1-3\delta^2))}, \\ r_0 D_2 &= h(r_0)^{\frac{\delta^2-1}{\delta^2+1}} \frac{1+\delta^2}{2(1+h(r_0)\delta^2)}, \\ r_0^{-1} K &= \frac{h(r_0)}{1+\delta^2} \frac{(2(1-\delta^2) - h(r_0)(1-3\delta^2))(2(3-\delta^2) - h(r_0)(1-3\delta^2)(4-h(r_0)(1-\delta^2)))}{2(3-\delta^2) - h(r_0)(1-3\delta^2)(3-\delta^2+2\delta^2 h(r_0))}, \\ r_0^{-1} \gamma &= \frac{16\pi^2 h(r_0)(1+\delta^2)(1+\delta^2 h(r_0))}{2(3-\delta^2) - h(r_0)(1-3\delta^2)(4-h(r_0)(1-\delta^2))}. \end{aligned} \quad (\text{I16})$$

The zero temperature limit of the Wiedemann-Franz ratio, and of the modified Wiedemann-Franz ratio (3.37), for these solutions is

$$\begin{aligned} \lim_{T \rightarrow 0} L &= \frac{8\pi^2 h(r_0) (1+\delta^2)^3 (4-h(r_0)(1-3\delta^2))}{(2(3-\delta^2) - h(r_0)(1-3\delta^2))(4-h(r_0)(1-\delta^2))^2}, \\ \lim_{T \rightarrow 0} L_\alpha &= -\mathcal{Q} \frac{\partial}{\partial\mathcal{Q}} \left(\frac{\mathcal{S}_0}{\mathcal{Q}} \right) \\ &= \frac{2\pi (1+\delta^2)^2 h(r_0)(4-h(r_0)(1-3\delta^2))}{\sqrt{(h(r_0)-1)(-3+\delta^2+h(r_0)(1-3\delta^2))(2(3-\delta^2) - h(r_0)(1-3\delta^2)(4-h(r_0)(1-\delta^2)))}}, \end{aligned} \quad (\text{I17})$$

which both vanish in the translationally invariant limit $m \rightarrow 0$. The zero temperature ‘equation of state’ $\mathcal{S}_0(\mathcal{Q})$ is given by the solution to

$$2\pi m^2 \left(\frac{2(1-\delta^2)\mathcal{S}_0 + (1+\delta^2)\sqrt{\mathcal{S}_0^2 + 4\pi^2(1-3\delta^2)\mathcal{Q}^2}}{\mathcal{S}_0(1-3\delta^2)} \right)^{\frac{\delta^2-1}{\delta^2+1}} + \sqrt{\mathcal{S}_0^2 + 4\pi^2(1-3\delta^2)\mathcal{Q}^2} - 2\mathcal{S}_0 = 0. \quad (\text{I18})$$

It naively appears that the relation (1.13) is true independently on the value of δ . In particular it seems to apply outside the range $0 \leq \delta < 1/\sqrt{3}$, where the solutions no longer have AdS_2 horizons. This is not the case – the condition (I5) only corresponds to the $T = 0$ limit of the system when δ is in this range. For example, for $\delta = 1/\sqrt{3}$ (when the near-horizon geometry is conformal to $\text{AdS}_2 \times \mathbb{R}^2$) the condition (I5) leads to an imaginary value of the chemical potential. The $T = 0$ limit of the $\delta = 1/\sqrt{3}$ solution is when $r_0 = 0$, and so the relation (1.13) is not true in this case.

2. Asymptotically AdS_5

There are an analogous class of solutions which are asymptotically AdS_5 [26]. In this case, the action is

$$S = \int d^5x \sqrt{-g} \left(\mathcal{R} - \frac{1}{2}(\partial\Phi)^2 + V(\Phi) - \frac{1}{2} \sum_{i=1}^3 (\partial\varphi_i)^2 - \frac{Z(\Phi)}{4} F_{\mu\nu} F^{\mu\nu} \right), \quad (\text{I19})$$

and the functions $V(\Phi)$ and $Z(\Phi)$ depend on a single parameter δ

$$V(\Phi) = \frac{18\delta^2(6\delta^2-1)}{(1+3\delta^2)^2} e^{-\frac{2\Phi}{3\delta}} + \frac{108\delta^2}{(1+3\delta^2)^2} e^{\frac{\Phi}{3\delta}(3\delta^2-1)} - \frac{6(3\delta^2-2)}{(1+3\delta^2)^2} e^{2\delta\Phi}, \quad Z(\Phi) = e^{-2\delta\Phi}. \quad (\text{I20})$$

The asymptotically AdS_5 solutions that break translational symmetry homogeneously are¹¹

$$\begin{aligned} ds^2 &= -f(r)h(r)^{-\frac{2}{1+3\delta^2}} dt^2 + \frac{dr^2}{f(r)h(r)^{-\frac{1}{1+3\delta^2}}} + r^2 h^{\frac{1}{1+3\delta^2}} d\vec{x}^2, & \varphi_i &= mx^i, \\ f(r) &= r^2 \left(h(r)^{\frac{3}{1+3\delta^2}} - \frac{r_0^4}{r^4} h(r_0)^{\frac{3}{1+3\delta^2}} \right) - \frac{m^2}{4} \left(1 - \frac{r_0^2}{r^2} \right), & h(r) &= 1 + \frac{Q}{r^2}, \\ A_t(r) &= \sqrt{\frac{3Q}{1+3\delta^2} \left(h(r_0)^{\frac{1-6\delta^2}{1+3\delta^2}} - \frac{m^2}{4r_0^2 h(r_0)} \right)} \frac{\left(1 - \frac{r_0^2}{r^2} \right)}{h(r)}, & \Phi &= -\frac{3\delta}{1+3\delta^2} \log h(r). \end{aligned} \quad (\text{I21})$$

¹¹ There are typos in the solution and thermodynamic properties in [26] which we have corrected here.

We will assume that both Q and r_0 are positive. The thermodynamic properties of these solutions are

$$\begin{aligned} T &= \frac{r_0}{2\pi} h(r_0)^{-\frac{3}{2(1+3\delta^2)}} \left(\frac{3 - (1 - 6\delta^2)h(r_0)}{(1 + 3\delta^2)} h(r_0)^{\frac{2-3\delta^2}{1+3\delta^2}} - \frac{m^2}{4r_0^2} \right), & \mathcal{S} &= 4\pi r_0^3 h(r_0)^{\frac{3}{2(1+3\delta^2)}}, \\ \mu &= \sqrt{\frac{3Q}{1 + 3\delta^2} \left(h(r_0)^{\frac{1-6\delta^2}{1+3\delta^2}} - \frac{m^2}{4r_0^2 h(r_0)} \right)}, & \mathcal{Q} &= 2\mu(Q + r_0^2). \end{aligned} \quad (\text{I22})$$

For $0 \leq \delta < 1/\sqrt{6}$, the $T = 0$ geometries are found by imposing the condition

$$\frac{3 - (1 - 6\delta^2)h(r_0)}{1 + 3\delta^2} h(r_0)^{\frac{2-3\delta^2}{1+3\delta^2}} = \frac{m^2}{4r_0^2}, \quad (\text{I23})$$

and we will restrict to these values of δ from now on. After changing coordinates to

$$r = r_0 + \epsilon h(r_0)^{\frac{1}{2(1+3\delta^2)}} \zeta, \quad t = \frac{\tau}{\epsilon}, \quad (\text{I24})$$

and taking the near-horizon limit $\epsilon \rightarrow 0$, we find a $T = 0$ charged $\text{AdS}_2 \times \mathbb{R}^3$ geometry

$$\begin{aligned} ds^2 &= \left(-\frac{\zeta^2}{\tilde{L}^2} d\tau^2 + \frac{\tilde{L}^2}{\zeta^2} d\zeta^2 \right) + r_0^2 h(r_0)^{\frac{1}{1+3\delta^2}} d\vec{x}^2 + O(\epsilon), \\ A_\tau &= \frac{\mathcal{E}}{\tilde{L}^2} \zeta + O(\epsilon), \quad \varphi_i = m x^i, \quad \Phi = -\frac{3\delta}{1 + 3\delta^2} \log(h(r_0)), \end{aligned} \quad (\text{I25})$$

with the AdS_2 radius of curvature

$$\tilde{L}^2 = \frac{(1 + 3\delta^2)^2 h(r_0)^{\frac{6\delta^2}{1+3\delta^2}}}{2(3(2 - 3\delta^2) - h(r_0)(1 - 6\delta^2)(6 - h(r_0)(2 - 3\delta^2)))}, \quad (\text{I26})$$

and the electric field

$$\mathcal{E} = (1 + 3\delta^2) \frac{\sqrt{3(h(r_0) - 1)(-2 + 3\delta^2 + h(r_0)(1 - 6\delta^2))}}{3(2 - 3\delta^2) - h(r_0)(1 - 6\delta^2)(6 - h(r_0)(2 - 3\delta^2))}. \quad (\text{I27})$$

The dc conductivities can be computed using the usual techniques [23, 80], and are given by

$$\sigma = \frac{\mathcal{S}}{4\pi g_{xx}(r_0)} Z(\Phi(r_0)) + \frac{4\pi \mathcal{Q}^2}{m^2 \mathcal{S}}, \quad \alpha = \frac{4\pi \mathcal{Q}}{m^2}, \quad \bar{\kappa} = \frac{4\pi \mathcal{S} T}{m^2}. \quad (\text{I28})$$

The $T \rightarrow 0$ limit of the Seebeck coefficient is then

$$S \equiv \frac{\alpha}{\sigma} = 2\pi(1 + 3\delta^2) \frac{\sqrt{3(h(r_0) - 1)(-2 + 3\delta^2 + h(r_0)(1 - 6\delta^2))}}{3(2 - 3\delta^2) - h(r_0)(1 - 6\delta^2)(6 - h(r_0)(2 - 3\delta^2))}. \quad (\text{I29})$$

Computing $(\partial \mathcal{S} / \partial \mathcal{Q})_T$ in a similar manner to the previous subsection, we find that equation (1.13) is true. This is further evidence that (1.13) is a consequence of the AdS_2 part of the near-horizon

geometry. The low energy correlators of the field theory states dual to these solutions will have the form (2.49) in the small T limit, with

$$\begin{aligned}
r_0 D_1 &= h(r_0)^{-\frac{3}{2(1+3\delta^2)}} \frac{3(2-3\delta^2) - h(r_0)(1-6\delta^2)(4-3\delta^2(2-3h(r_0)))}{2(3(1-3\delta^2) - 2h(r_0)(1-6\delta^2))(3-h(r_0)(1-6\delta^2))}, \\
r_0 D_2 &= h(r_0)^{\frac{-1+6\delta^2}{2(1+3\delta^2)}} \frac{1+3\delta^2}{3(1+3h(r_0)\delta^2)}, \\
r_0^{-2} K &= \frac{2h(r_0)}{1+3\delta^2} \frac{(3(1-3\delta^2) - 2h(r_0)(1-6\delta^2))(3(2-3\delta^2) - h(r_0)(1-6\delta^2)(6-h(r_0)(2-3\delta^2)))}{3(2-3\delta^2) - h(r_0)(1-6\delta^2)(2(2-3\delta^2) + 9h(r_0)\delta^2)}, \\
r_0^{-2} \gamma &= \frac{12\pi^2 h(r_0)(1+3\delta^2)(1+3h(r_0)\delta^2)}{3(2-3\delta^2) - h(r_0)(1-6\delta^2)(6-h(r_0)(2-3\delta^2))}.
\end{aligned} \tag{I30}$$

It is straightforward to calculate the Wiedemann-Franz ratio L , and the modified ratio L_α , which have the zero temperature values

$$\begin{aligned}
\lim_{T \rightarrow 0} L &= \frac{4\pi^2 h(r_0)(1+3\delta^2)^3 (3-h(r_0)(1-6\delta^2))}{(3(2-3\delta^2) - h(r_0)(1-6\delta^2)(6-h(r_0)(2-3\delta^2)))^2}, \\
\lim_{T \rightarrow 0} L_\alpha &= -\mathcal{Q} \frac{\partial}{\partial \mathcal{Q}} \left(\frac{\mathcal{S}_0}{\mathcal{Q}} \right) \\
&= \frac{2\pi (1+3\delta^2)^2 h(r_0)(3-h(r_0)(1-6\delta^2))}{\sqrt{3(h(r_0)-1)(-2+3\delta^2+h(r_0)(1-6\delta^2))(3(2-3\delta^2) - h(r_0)(1-6\delta^2)(6-h(r_0)(2-3\delta^2)))}}.
\end{aligned} \tag{I31}$$

for these solutions. These vanish when translational invariance is restored ($m \rightarrow 0$). The zero temperature ‘equation of state’ $\mathcal{S}_0(\mathcal{Q})$ is given by the solution to the equation

$$\begin{aligned}
&6^{\frac{3}{1+3\delta^2}} \pi m^3 \mathcal{S}_0^{1/2} \left(\frac{9\mathcal{S}_0(1-3\delta^2) + (1+3\delta^2)\sqrt{9\mathcal{S}_0^2 + 48\pi^2(1-6\delta^2)\mathcal{Q}^2}}{\mathcal{S}_0(1-6\delta^2)} \right)^{\frac{3(3\delta^2-1)}{2(3\delta^2+1)}} \\
&- 2 \left(9\mathcal{S}_0 - \sqrt{9\mathcal{S}_0^2 + 48\pi^2(1-6\delta^2)\mathcal{Q}^2} \right)^{3/2} = 0.
\end{aligned} \tag{I32}$$

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